Topos Formulation of Quantum Theory

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1 Philosophy Behind The Topos Approach To Quantum Theory

1) Each physical system S is associated with a local language, $\mathcal{L}(S)$.

2) The application of the theory-type (for example, classical physics, or quantum physics) to S involves finding a representation of $\mathcal{L}(S)$ in an appropriate topos τ within whose framework the theory, as applied to S, is to be formulated and interpreted.

3) Each Topos has an internal language associated with it, thus a theory of physics is equivalent to finding a translation/representation of $\mathcal{L}(S)$ into the internal language of that topos.

Thus the topos approach consists of:

1) First understanding at a fundamental level what a theory of physics and associated conceptual framework should look like.

2) Defining a theory of physics as an interplay between a theory-type, a system with associated internal language, and the appropriate topos in which to represent such a language.

2) Applying these insights to various system-language-theory-type.

2 What is Topos Theory?

- A category is a collection of objects and a collection of 'maps' between these objects.

  The best-known example is *Sets*. But.......

- A topos is a category which is similar to *Sets*: fundamental mathematical properties (disjoint union, Cartesian product, etc) have a topos analogue. In particular
  - **Sub-object classifier** Ω: Generalises the set $\{0, 1\}$ of truth-values in the category *Sets*.
  - Collection of all sub-objects of any object forms a **Heyting algebra**:

    A distributive algebra for which $S \vee \neg S \leq 1$. An internal logic, analogue to Boolean algebra in *Sets*

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1 Why Topos Theory?

- Kocken-Specher theorem $\Longrightarrow$ non-realist interpretation of quantum theory

  For quantum cosmology, need a reformulation of quantum theory which is 'more realist'. Isham, Butterfield & Döring: can be done through *topos theory*.

- Reformulate quantum theory to make it 'look like' classical physics:
  - Classical physics uses *Sets* as its mathematical structure.
    A topos is a category which 'looks like' *Sets*.
  - Logic of subsets in *Sets* is Boolean logic.
    Logic of subsets in a topos is a distributive logic

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2.2 Which Topos?

- Need for *contexts* comes from K-S theorem: only within *abelian subalgebras* of $\mathcal{B}(\mathcal{H})$ can quantum theory 'look like' classical theory. *Contexts* form 'classical snapshots' from whose perspectives quantum theory can be displayed..
  - The set of abelian subalgebras, $\mathcal{V}(\mathcal{H})$, forms a category (just a poset) under subset inclusion: $i_{V'V} : V^{'} \subseteq V$

    i.e. consider all contexts at the same time!
  - **Example**: $V^{''} = V \cap V^{'} \neq \emptyset$ then $\exists$ the inclusion maps $i_{V''V}$ and $i_{V''V'}$, therefore it is possible to 'relate' V and $V^{'}$.
- Such sub-algebra are context in which the theory can be viewed *locally* from a classical perspective. However the *global* information is retained by the categorical structure of $\mathcal{V}(\mathcal{H})$

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2.3 Topos of Presheaves

Topos of presheaves over the category of abelian subalgebras : $Sets^{\mathcal{V}(\mathcal{H})^{\mathrm{op}}}$.

Let $\mathcal{C}, \mathcal{D}$ be categories. Then a presheaf is an assignment to each $\mathcal{D}$-object A of a $\mathcal{C}$-object $X(A)$, and to each $\mathcal{D}$-arrow $f : A \to B$ a $\mathcal{C}$-arrow $X(f) : X(B) \to X(A)$ such that:

- $X(1_A) = 1_{X(A)}$

- $X(f \circ g) = X(g) \circ X(f)$ for any $g : C \to A$

3. The Isham-Doering scheme

3.1 The State Object

State spaces in physics

1. *Classical physics*: Physical quantity A represented $f_A : \mathcal{S} \rightarrow \mathcal{R}$.
2. *Quantum physics*: Physical quantity A represented $\hat{A} : \mathcal{H} \rightarrow \mathcal{H}$.
3. *Topos physics*: *Spectral presheaf* $\underline{\Sigma} : \mathcal{V}(\mathcal{H}) \rightarrow \mathit{Set}$ such that

   $V \mapsto \underline{\Sigma}_V := \{\text{simultaneous eigenvalues of } V\}$; i.e., the possible values of the physical quantities in V.

   Given $\hat{A}$, then $f_{\hat{A}} : \underline{\Sigma} \rightarrow \underline{\mathcal{R}}$!

$\underline{\Sigma}$ is the '**state object**' of the theory.

3.2 Propositions

Propositions

1. *Classical physics*:

$$\text{“}A \in \Delta\text{”} \to f_A^{-1}(\Delta) = \{s \in \mathcal{S} | f_A(s) \in \Delta\} \subseteq \mathcal{S}$$

2. *Quantum theory*:

$$\hat{P} = \hat{E}[A \in \Delta] \in P(\mathcal{H})$$

3. *Topos physics*: Need to identify $\hat{P}$ with sub-object of $\underline{\Sigma}$; i.e., for each V need subset of $\underline{\Sigma}_V$; i.e., a *projection operator* in V.

3.3 Daseinisation

- **'Daseinisation':**

$$\delta : P(\mathcal{H}) \to P(V)$$
$$P \mapsto \delta(\hat{P})_V$$

where $\delta(\hat{P})_V := \bigwedge\{\hat{\alpha} \in P(V) | \hat{\alpha} \geq \hat{P}\}$: the 'best' approximation of $\hat{P}$ (from above) by projectors in V.

- **Relation to $Sub(\underline{\Sigma})$:**

Any projector in V gives a subset of $\underline{\Sigma}_V$. Therefore get map, for each V, $\hat{P} \mapsto \underline{\delta(\hat{P})}_V$.

Can show that this corresponds to $\delta : P(\mathcal{H}) \to Sub(\underline{\Sigma})$!

Thus δ maps propositions about a quantum system to a *distributive* lattice in a *contextual* manner.

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3.4 States

- **States in classical physics**

  In classical physics a microstate is a point in the state space.

- **States in the topos formulation of quantum theory**
  - $\underline{\Sigma}$ has no 'points' $\Longrightarrow$ Equivalent to the K-S theorem
  - Topos analogues of a state is a (non-point!) sub-object of the state object $\underline{\Sigma}$:

  Pseudo-state: $\underline{\mathfrak{w}}^{|\psi\rangle} := \underline{\delta(|\psi\rangle\langle\psi|)} \subseteq \underline{\Sigma}$,

$$\delta(\,|\psi\rangle\langle\psi|)_V := \bigwedge\{\hat{\alpha} \in P(V) \mid |\psi\rangle\langle\psi| \leq \hat{\alpha}\}$$

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3.5 Truth Values

- **Truth values in classical physics**

  A proposition $Q \subseteq \mathcal{S}$ is true in a state s iff $s \in Q$.
  This is equivalent to $\{s\} \subseteq Q$.

- **Truth values in the topos formulation of quantum theory**
  - In topos quantum theory, we define the proposition $\underline{\delta(\hat{P})}$ to be 'totally true' given the pseudo state $\underline{\mathfrak{w}}^{|\psi\rangle}$ iff $\underline{\mathfrak{w}}^{|\psi\rangle} \subseteq \underline{\delta(\hat{P})}$. This means that, for all V, $\mathfrak{w}_V^{|\psi\rangle} \leq \delta(\hat{P})_V$.
  - *However*, in a topos a proposition can be 'partially true' using 'contextual truth values'. At stage V we define

$$v(\underline{\mathfrak{w}}^{|\psi\rangle} \subseteq \underline{\delta(\hat{P})})_V := \{V' \subseteq V \mid \underline{\mathfrak{w}}_{V'}^{|\psi\rangle} \subseteq \underline{\delta(\hat{P})}_{V'}\}$$
$$= \{V' \subseteq V \mid \langle\psi|\delta(\hat{P})_{V'}|\psi\rangle = 1\}$$

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3.5 Truth values

– This is called a *sieve* on V.

The collection, $\underline{\Omega}_V$, of all sieves on V forms a **Heyting algebra**!

– For varying V such truth values form a global section $\Gamma\underline{\Omega}$ of the sub-object classifier $\underline{\Omega}$.

The set $\Gamma\underline{\Omega}$ is itself a Heyting algebra!

– Thus we have a Heyting algebra of propositions *and* a Heyting algebra of 'generalised' truth values!

3.6 Notion of Contextuality

Notion of contextuality not defined as ascribing 'reality' to only some commutative subset of physical quantities: value given to a physical quantity A depends on some context in which A is to be considered and cannot be part of a global assignment of values.

In the topos schema notion of 'generalised' valuations that are defined globally on all propositions about values of physical quantities.

However, the price of global existence is, (i) the truth-value of a proposition $A \in \Delta$ belongs to a logical structure that is larger than $\{0, 1\}$ (multivalued logic); and (ii) these target-logics, and truth values, are context-dependent

4 Conclusion and Outlook

- Symbols types: $\underline{\Sigma}$; $\mathcal{R}$; $\underline{\Omega}$; $\underline{\delta\hat{P}}, \underline{\mathfrak{w}}^{|\psi\rangle} \in P(\underline{\Sigma})$. Function symbol $\underline{\Sigma} \to \mathcal{R}$ are represented for classical physics in Sets for Quantum theory in the topos $Sets^{\mathcal{V}(\mathcal{H})^{\mathrm{op}}}$.
- Managed to re-express quantum theory in a more realist way by utilising the mathematical framework of Topos theory, in particular the topos of presheaves. This enabled to interpret the symbol types and the function symbols in analogy with classical physics.
- In this context it is possible to meaningfully assign truth values to any proposition, however the collection of truth values is larger than just $\{\text{true}, \text{false}\}$. We get a multivalued logic (intuitionistic logic)

3.2 Example of The State Object

Consider the Hilbert space $\mathcal{H} = \mathbb{C}^3$ with $B(\mathbb{C}^3)$ algebra of bounded linear operators.

Define the abelian subalgebra $V := lin_{\mathbb{C}}(\hat{P}_1, \hat{P}_2, \hat{P}_3)$

Spectrum of V

$$\underline{\Sigma}_V := \{\lambda_1, \lambda_2, \lambda_3\} \text{ such that } \lambda_i(\hat{P}_j) = \delta_{ij}$$

Consider the subalgebra $V' := lin_{\mathbb{C}}(\hat{P}_1, \hat{P}_2 + \hat{P}_3) \subseteq V$.

Spectrum of V'

$$\underline{\Sigma}_{V'} := \{\lambda_1', \lambda_2'\}$$

$$\lambda_1'(\hat{P}_1) = 1 \, ; \lambda_1'(\hat{P}_2 + \hat{P}_3) = 0 \, ; \lambda_2'(\hat{P}_1) = 0 \, ; \lambda_2'(\hat{P}_2 + \hat{P}_3) = 1$$

Morphisms $\underline{\Sigma}_{V'V}$

$$\underline{\Sigma}_V \to \underline{\Sigma}_{V'}$$

$$\lambda_1 \mapsto \lambda_1' \, ; \lambda_2 \mapsto \lambda_2' \, ; \lambda_3 \mapsto \lambda_2'$$

It is interesting to note that given the isomorphisms $I : [0,1]^h \rightarrow \Gamma(\underline{\Omega}^{(0,1]})$ and the map $\epsilon^\mu : Sub_{Set^{(0,1]}}(\underline{X}) \rightarrow \Gamma\underline{\Omega}^{(0,1]}$ we have the following commutative diagram

$$\begin{array}{ccc} Sub(X) & \xrightarrow{\mu} & [0,1]^h \\ \Big\downarrow f & & \Big\downarrow I \\ Sub_{Set^{(0,1]}}(\underline{X}) & \xrightarrow{\epsilon^\mu} & \Gamma(\underline{\Omega}^{(0,1]}) \end{array}$$

3.7 Example for Classical Probability Theory

- Probabilities get their values in the set $[0, 1]$
- Possible to turn the set $[0, 1]$ into a Heyting algebra: $[0, 1]^h$. This can be done for any totally ordered set.
- Possible to define an isomorphisms between the following Heyting algebras:

  $$[0, 1]^h \cong \Gamma(\underline{\Omega}^{\tau})$$

  where $\tau = Set^{[0,1]^{op}}$.

  The correct topos to use to represent classical probabilities is $\tau = Set^{(0,1]^{op}}$

- Consider a space X with probability measure $\mu : Sub(X) \to [0,1]$.
  Translate this setting into the language of the topos of presheaves $\tau = Set^{(0,1]^{op}}$.
- Map each element to the constant presheaf: $X \to \underline{X}$.
  $\underline{X}$ is such that for all $r \in (0,1]$, $\underline{X}(r) = X$.
  In Set propositions are identified with elements in $Sub(X)$.
  In $Set^{(0,1]^{op}}$ propositions are identified with elements of $Sub_\tau(\underline{X})$.
- Truth Object: $\underline{T}^\mu_r := \{S \subseteq X | \mu(S) \geq r\}$ for all $r \in (0,1]$.

States $\Leftrightarrow$ truth objects

Quantum case

$$\begin{aligned}\mathcal{T}_V^{|\psi\rangle} &:= \{\hat{A} \in P(V) | Prob(\hat{A}|\psi\rangle = 1\} \\ &= \{\hat{A} \in P(V) | \langle\psi|\hat{A}\psi\rangle = 1\}\end{aligned}$$

Pseudo-state: $\underline{\mathfrak{w}}^{|\psi\rangle} := \underline{\delta(|\psi\rangle\langle\psi|)} \subseteq \underline{\Sigma}$,

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can be shown that

$$v(\underline{\mathfrak{w}}^{|\psi\rangle} \subseteq \delta(\hat{P})) \cong v(\delta(\hat{P}) \in \mathcal{T}^{|\psi\rangle})$$