INTRODUCTION

"Useful as it is under everyday circumstances to say that the world exists "out there" independent of us, that view can no longer be upheld. There is a strange sense in which this is a "participating universe"", Wheeler (1983).

The above statement reveals the radical difference that exists between the view of the world given by Quantum Mechanics and the view given by Classical Physics. In fact, the existence of an "objective external world", which is postulated by Classical Physics, seems to be rejected by Quantum Mechanics. The cause of this interpretative differences between the two theories can be traced back to the different algebras used to relate propositions[1]. Precisely the propositions in Classical Physics form a Boolean algebra, while propositions in Quantum Mechanics form a non-Boolean algebra. This feature of Quantum Mechanics entails that properties can not be said to be possessed by a system, denying, in such a way, the existence of an independent "outside world". This view seem ulteriorly confirmed by the Kochen-Specker theorem that asserts the impossibility of evaluating propositions regarding values possessed by physical entities (represented by projection operators), such that their truth values belong to the set $\{1, 2\}$, therefore depriving of meaning any statement regarding a state of affairs of a system, since generally speaking, a statement is said to be meaningful if its validity can be assessed. Do we then have to accept that Quantum theory is a non-realist theory and therefore regard any statement about states of affair of a system as meaningless? Following the steps of C.J.Isham and J. Butterfield, I will show that it is possibility to retaining some realist flavor in Quantum Theory by changing the logical structure with which propositions about the values

[1]Propositions are defined as statements regarding properties of a given system.

of physical quantities are handled. In particular I will introduce a new kind of valuation for quantum quantities which is defined on all operators, so that it will be possible to assign truth values to Quantum Proposition. This new valuation will be defined using Topos Theory, and it will be such that the truth values become multi-valued and contextual, in agreement with the mathematical formalism of Quantum Theory. The idea behind the definition of this new valuations is the given by the realization that, although the Kochen-Specker Theorem prohibits to assign truth values to propositions , it nevertheless allows the possibility of assigning truth values to generalized propositions Therefore, by adopting generalized propositions as the domain of applicability of the valuation function, we obtain a situation in which it is meaningful to assign truth values.

MATHEMATICAL PRELIMINARIES

Topos

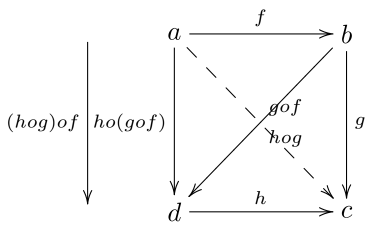
Generally speaking a **Topos** is a Category with some extra properties.

Definition 1 *A **category** consists of two things:*

1. *a collection of objects*
2. *a collection of morphisms between these objects such that the following conditions hold:*

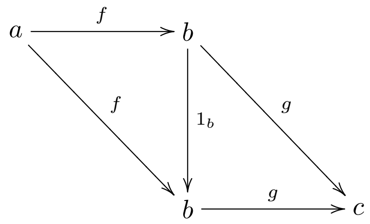
- ***composition condition**: given two morphisms* $f : a \to b$ *and* $g : b \to c$ *with dom g=cod f then there exists the composite map* $gof : a \to c$
- ***associative law**: given* $a \xrightarrow{f} b \xrightarrow{g} c$ *then (h o(g o f))=((h o g)o f) i.e. the following diagram commutes*

Diagram 1



- ***identity law** : for any object b in the category there exists a morphism* $1_b : b \to b$ *called identity arrow such that, given any other two morphisms* $f : a \to b$ *and* $g : b \to c$ *we then have* $1_b o f = f$ *and* $g o 1_b = g$*, i.e. the following diagram commutes*

Diagram 2

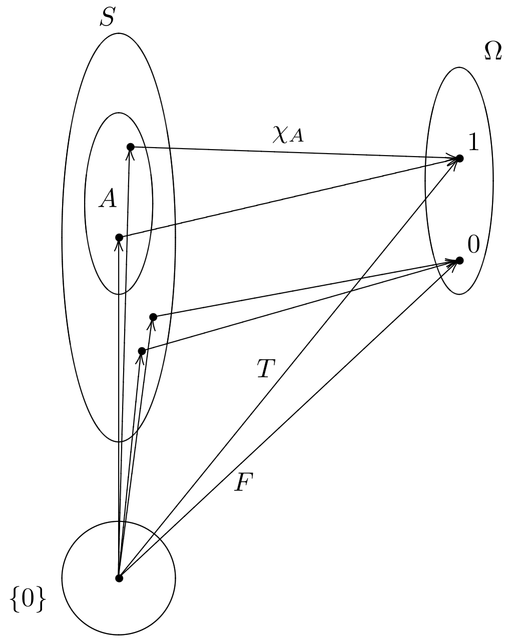


The property of a Topos which is relevant to our discussion is the **Subobject classifier** Ω (also called the truth value object)

Essentially what a subobject classifier does is to define subobjects according to which elements belong to them. Specifically it assigns the values true to elements belonging to the subset in question, and false to elements not belonging to the subset. So one could say that a subobject classifier in Topos theory is a generalization of the idea, in set theory, of a characteristic function.

Note: that Ω itself is an object of the Topos.

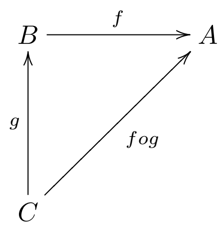
The way in which the subobject classifier works can be easily seen with the aid of the following diagram which represents the subobject classifier for Sets where the value true is given by the arrow $\{0\} \rightarrow 1$



Where $\{0\} = 1$,T=*true* and F=*false*

In Topos theory, though, truth values are not represented by the set $\{0, 1\}$, but are instead represented by sieves. Let us define what a sieve is. **Sieve**

Definition 2 *A* ***sieve*** *on an object* $A \in \mathscr{C}$ *is a collection* S *of morphisms in* $\mathscr{C}$ *whose codomain is A and such that, if* $f : B \to A \in S$ *then, given any morphisms* $g : C \to B$ *we have* $fog \in S$*, i.e.* S *is closed under left composition:*



For example in a poset a sieve is an upper set. Specifically, given a poset C, a sieve on $p \in C$ is any subset S of C such that if $r \in S$ the 1) $p \leq r$ 2) $r^{‘} \in S \ \ \forall r \leq r^{‘}$.

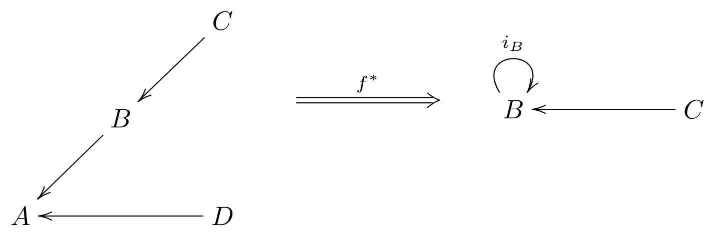
A map $\Omega_{qp} : \Omega_p \to \Omega_q$ between sieves exists iff $p \leq q$ and given $S \in \Omega_q$ then

$$\Omega_{qp}(S) := \uparrow p \cap S$$

where $\uparrow p := \{r \in C | p \leq r\}$

An important property of sieves is the following: if $f : B \to A$ belongs to S which is a sieve on A, then the pullback of S by f determines a principal sieve on B, i.e.

$$f^*(S) := \{h : C \to B | foh \in S\} = \{h : C \to B\} = \downarrow B$$



The principal sieve of an object A, denoted by $\downarrow A$, is the sieve that contains the identity morphism of A therefore it is the biggest sieve on A.

An important properties of sieves is that the set of sieves defined on an object forms an Heyting algebra, with partial ordering given by subset inclusion.

Definition 3 *A* ***Heyting Algebra H*** *is a* ***relative pseudo complemented distributive lattice****.*

The property of being **relative pseudo complemented lattice** means that for any two elements $S_1, S_2 \in \mathbf{H}$ there exist a third element $S_3 \in \mathbf{H}$, such that:

1. $S_1 \cap S_3 \subseteq S_2$

2. $\forall S \in \mathbf{H} \quad S \subseteq S_3 \quad iff \quad S_1 \cap S \subseteq S_2$

where S_3 is defined as the *pseudo complement* of S_1 relative to S_2 i.e., the greatest element of the set $\{S : S_1 \cap S \subseteq S_2\}$, and it is denoted as $S_1 \Rightarrow S_2$.

A particular feature of the Heyting algebra is the negation operation.

The negation of an element S is defined to be the pseudo-complement of S i.e. $\neg S := S \Rightarrow 0$ Therefore we can write

$$\neg S := \{f : B \to A | \forall g : C \to B, fog \notin S\}$$

The above equation entails that $\neg S$ is the least upper bound of the set $\{x : S \cap x = 0\}$, i.e. the biggest set that does not contain any element of S. From the above definition of negation operation it follows that the Heyting algebra does not satisfy the law of excluded middle [**?**], i.e. given any element S of an Heyting algebra we have the following relation: $S \vee \neg S \leq 1$.

Proof 1 *Let us consider* $S \vee \neg S = S \cup \neg S$*, this represents the least upper bound of* S *and* $\neg S$ *therefore, given any other element* S_1 *in the Heyting algebra such that* $S \leq S_1$ *and* $\neg S \leq S_1$*, then* $S \vee \neg S \leq S_1$*. But since for any* S *we have* $S \leq 1$ *and* $\neg S \leq 1$ *it follows that* $S \vee \neg S \leq 1$*.*

QUANTUM PROPOSITIONS

Propositions in Quantum Mechanics

In Quantum Mechanics propositions of the form $(A \in \Delta)$ are represented by spectral projectors $\hat{E}[A \in \Delta] = \hat{P}_{A\in\Delta}$ Therefore propositions are identified with subspaces $\mathcal{H}_{\hat{P}_{A\in\Delta}} = \{\vec{\Psi} \in \mathcal{H} | \hat{P}_{A\in\Delta}\vec{\Psi} = \vec{\Psi}\}$ (range of P) of the Hilbert space $\mathcal{H}$ for which they are true.

Note: $\hat{P}_{A\in\Delta}$ represents equivalent classes of Propositions.

Coarse-graining of a Quantum Proposition

Generally speaking a coarse graining of a proposition is a generalization of that proposition. Therefore, we then say that "$f(A) \in f(\Delta)$" is a coarse-grainig of "$A \in \Delta$" i.e. " "$A \in \Delta$" implies "$f(A) \in f(\Delta)$"", but the converse is not true. In terms of spectral projectors the coarse-graining relation is defined as follows: $\hat{P}_{A\in\Delta} \leq \hat{P}_{f(A)\in f(\Delta)}$

For example if we knew that A=2 it follows that $A^2 = 4$, but if we only knew that $A^2 = 4$ all we could say about A is that its value could be either 2 or -2. The weakening of a proposition can occurs whenever the function f is many to one. In terms of spectral algebras the relation of coarse graining is given in terms of subset inclusion. In fact we say that the spectral algebra $W_{f(A)}$ of the operator $f(\hat{A})$ is a coarse graining of the spectral algebra W_A of the operator A iff $W_{f(A)} = W_B \subseteq W_A$.

This implies(means) that every element in W_B can be written as the logical "or" of disjoint elements in W_A, hence W_B is the coarse-grainig of W_A.

REQUIREMENTS FOR VALUATIONS IN QUANTUM MECHANICS

- **Contextuality**: truth values of a proposition depend on what other propositions are being evaluated at the same time. $\Longrightarrow$ required by the quantum mechanical formalism.
- **Multi-valued** $\Longrightarrow$ required by the Kochen-Specker theorem
- **Space of truth values is equipped with some kind of logical structure** $\Longrightarrow$ the logical structure of propositions being evaluated has to be mapped to the logical structure of the space of truth values.
- **Truth value of propositions "$A \in \Delta$" is given in terms of its coarse-grainings "$f(A) \in f(\Delta)$"** $\Longrightarrow$ since Kochen-Specker theory admits such truth values.

QUANTUM VALUATIONS IN THE LANGUAGE OF TOPOS THEORY

The first step in constructing a quantum valuation in the language of Topos theory is to turn $\mathcal{O}$ (set of all bounded self adjoint operators) in a category, where we define:

- elements are represented by self adjoint operators
- arrows $f_{\mathcal{O}} : \hat{B} \rightarrow \hat{A}$ exist iff given a Borel function $f : \sigma(\hat{A}) \rightarrow \mathbb{R}$ then $\hat{B} = f(\hat{A})$.

Coarse-graining in Quantum Mechanics

Since form K-S theorem it follows that truth values of propositions are given in terms of their coarse grainings, we need to give a mathematical definition of a coarse graining. Specifically we define the **coarse-graining** preshaef G on $\mathcal{O}$, as a contravariant functor $G : \mathcal{O} \rightarrow Set$ such that

- $G(\hat{A}) = W_A$=spectral algebra of A
- There exists a morphism $G(f_{\mathcal{O}}) : W_A \rightarrow W_B$ iff $\hat{B} = f(\hat{A})$ ($W_B \subseteq W_A$) such that $G(f_{\mathcal{O}})(\hat{P}_{A\in\Delta}) = \hat{P}_{f(A)\in f(\Delta)}$

In terms of the lattice of operators, the action of $G(f_{\mathcal{O}})$ is to move from one operator to the one immediately above it.

Definition 4 *A* ***generalized valuation*** *on Quantum propositions of the form $A \in \Delta$ (where Δ is a Borel subset of $\sigma(\hat{A})$) is a map $v : \mathcal{O} \rightarrow \Omega$ such that to each element $\hat{A} \in \mathcal{O}$ it assigns a sieve $v(A \in \Delta) \in \Omega(\hat{A})$ on $\hat{A}$, where $\hat{A}$ is called the stage of truth.*

v is such that, given any Borel function $h : \sigma(\hat{A}) \to \mathbb{R}$ we have

$$v_B(h(A) \in h(\Delta)) = h^*_{\mathcal{O}}(v_A(A \in \Delta)) \tag{1}$$

where $h_{\mathcal{O}} := h(\hat{A}) \to \hat{A}$ and $h^*_{\mathcal{O}}$ is the pullback of the sieve $v(A \in \Delta)$ on $\hat{A}$ to the sieve on $h(\hat{A})$.

What equation 1 means is that the truth value of propositions of the form "$A \in \Delta$" is determined in terms of the coarse-graining proposition "$h(A) \in h(\Delta)$" such that they are evaluated as being totally true at their own "stage of truth".

Example of Valuation

Given a quantum state $|\Psi\rangle \in \mathcal{H}$, the associated value for a proposition $A \in \Delta$ is:

$$v^{|\Psi\rangle}(A \in \Delta) := \{f_{\mathcal{O}} : \hat{B} \to \hat{A} | \hat{E}[B \in f(\Delta)]\Psi = \Psi\}$$
$$:= \{f_{\mathcal{O}} : \hat{B} \to \hat{A} | \hat{P}_{B \in f(\Delta)}\Psi = \Psi\}$$

(truth value of proposition $(A \in \Delta)$ is given in terms of a set of functions such that $|\Psi\rangle$ is in the range of the coarse grained projector with respect to those functions) Since $f_{\mathcal{O}} : \hat{B} \to \hat{A}$ exists iff $\hat{B} = f(\hat{A})$ What the above equation uncovers is that $f(\hat{A})$ is the truth value of $A \in \Delta$ iff $|\Psi\rangle$ is in the range of the spectral projector $\hat{E}[B \in f(\Delta)]$, where $\hat{E}[A \in \Delta] \leq \hat{E}[B \in f(\Delta)]$ in the lattice of projectors in the Hilbert space.

In other words, the value of a proposition is given in terms of its coarse grainings such that they are true with respect to the eigenvalue-eigenstate link

Worked example of a valuation

We will now give an example, taken for isham1999, of how a valuation, whose truth values are sieves, is obtained. Let us consider a spin 1 system. In particular we consider the operator $\hat{S}_z$ defined as

$$\hat{S}_z = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and the coarse graining of this operator defined by

$$\hat{S}_z^2 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Now given a state $\psi = (1, 0, 1)$ which is an eigenstate of $\hat{S}_z^2$ but not of $\hat{S}_z$, we want to obtain an expression for the truth value of the proposition $\hat{S}_z = 1$ given ψ. Thus we want to obtain the analog of the general formula for the truth value of a proposition namely the analog for our case of

$$\mathcal{V}^{|\psi\rangle}(A \in \Delta) = \{f_{\mathcal{O}} : \hat{B} \rightarrow \hat{A} | \hat{P}_{f(\hat{A} \in f(\Delta)} |\psi\rangle = |\psi\rangle\}$$

So what is required is the set of functions $f_{\mathcal{O}}$ which coarse grain the operator $\hat{S}_z$. These are straight forward to define since

$$\hat{S}_z^3 = \frac{1}{2}\hat{S}_z$$
$$\hat{S}_z^4 = \frac{1}{2}\hat{S}_z^2$$

thus generalizing we have that for n=odd $S_z^n = \frac{1}{2}\hat{S}_z$ while for n=even $S_z^n = \frac{1}{2}\hat{S}_z^2$. Thus a general function on $\hat{S}_z$ will be of the form

$f(\hat{S}_z) = t\hat{1} + k\hat{S}_z + r\hat{S}_z^2$

where t,k, and r are some real numbers.

Now we know that $|\psi\rangle$ is not an eigenstate of $\hat{S}_z$ thus $f(\hat{S}_z)|\psi\rangle = \lambda|\psi\rangle$ iff k=0. Therefore the set of functions $\{t\hat{1} + r\hat{S}_z^2\}$ define coarse-graining operators of $\hat{S}_z$

such that $|\psi\rangle$ is an eigenstate of them. Thus the analogue of the sieve functions will be

$$f_{\mathcal{O}} : t\hat{1} + r\hat{S}_z^2 \rightarrow \hat{S}_z$$

which is related to the Borel function such that

$$\begin{aligned} f : &\sigma(\hat{S}_z) \rightarrow \mathbb{R} \\ &\lambda \mapsto t + r\lambda^2 \end{aligned}$$

Thus we can now define the truth values for propositions regarding the possible values of the operator $\hat{S}_z$ given the state $|\psi\rangle$ namely

$$\begin{aligned} \mathcal{V}^{|\psi\rangle}(\hat{S}_z = 1) &= \{f_{\mathcal{O}} : t\hat{1} + r\hat{S}_z^2 \rightarrow \hat{S}_z | t, r \in \mathbb{R}\} \\ \mathcal{V}^{|\psi\rangle}(\hat{S}_z = -1) &= \{f_{\mathcal{O}} : t\hat{1} + r\hat{S}_z^2 \rightarrow \hat{S}_z | t, r \in \mathbb{R}\} \end{aligned}$$

CONSEQUENCES OF DEFINITION OF VALUATIONS

- **Contextuality** $\Longrightarrow$ The algebra $\Omega(A)$ to which $v(A \in \Delta)$ belongs, represents the Heyting algebra formed by the set of sieves on $\hat{A}$ therefore, it depends on the context $\hat{A}$.

- **Multi-valued truth values** $\Longrightarrow$ by identifying truth values with set of sieves, truthfulness of a proposition is directly proportionate to the size of the sieve (the bigger the sieve the more true the proposition $A \in \Delta$ is).

- **Relation between logic of proposition being evaluated and logic of the space of truth values** $\Longrightarrow$ Since the set of sieves defined on an object defines a Heyting algebra, it is possible to define the logical connectives in term of set operations. In particular we have:

  $$S_1 \wedge S_2 := S_1 \cap S_2 = \text{greatest lower bound}$$

  $$S_1 \vee S_2 := S_1 \cup S_2 = \text{least upper bound}$$

  $$S_1 \Rightarrow S_2 := \{f : B \to A | \forall g : C \to B fog \in S_1 \Rightarrow fog \in S_2\}$$

  $$\neg S := (S \Rightarrow 0) = \{f : B \to A | \forall g : C \to B fog \notin S\}$$

  Since a Heyting algebra is distributive, it can be used as a deductive system of reasoning. Moreover, since the Heyting structure associated with the set of sieves on an object derives from Ω, which in turn is fixed by the structure of the Topos it belongs to, it follows that the Heyting structure uniquely derives form the mathematical formalism itself, it is not arbitrarily chosen a posterior.

It is easy to see that the consequences of adopting a sieved-valued valuation coincide with the requirements of a valuation in Quantum mechanics.

CONCLUSIONS

In this discussion we have show that, although the Kochen-Specker theorem seems to give Quantum Theory an anti-realist interpretation, it is possible to define a sieved-valued valuation, such that the truth values it assigns to quantum propositions are multi-valued and contextual . These features are desirable since they agree with the mathematical formalism of Quantum Theory. Moreover this approach shows us that by agreeing on more general propositions, we obtain a situation in which it is meaningful to ask questions about stat of affairs of quantum systems.

PROBLEMS TO BE SOLVED

There are still a number of unresolved problems in this approach, namely:

- How to translate the uncertainty relation in the language of Topos Theory.
- How quantum entanglement is reflected in the value assignment given by the sieved-valued valuation.
- what is the mathematical status of the valuations v^p generated by the mixed state p in the quantum system.

how to recover Boolean valuations for the classical limit.

TOPOS THEORY IN QUANTUM THEORY AND QUANTUM GRAVITY

- Sieved-valued valuations are utilized in the Consistent History approach to Quantum Gravity, where the propositions being evaluated regard the history of the system rather than certain properties at a fixed time. In this approach Topos lorepresents the Heyting algebra formed by the set of sieves ongic is used to interpret the probabilistic predictions of Quantum Theory in the context in which all histories are handled at once. (kind of many worlds approach)

  "Chris Isham Imperial college".

- Transcendental Quantisation: In order to find a theory of Quantum gravity one needs to redefine the foundations of mathematics itself since the latter is based on set theory, which is essentially classical. Therefore to reach a Quantum Theory of Gravity one should find "quantum analogs" of the categories of classical mathematics. One approach is to start with Quantum Logic and derive an analogous non-Boolean set theory.

  "Chris Isham Imperial college"

- Replace the notion of the Continuum with an alternative conception of space and time using Topos Theory $\Longrightarrow$ no a priory reason to use continuum in our physical theories.

  R. D. Sorkin, **Ioannis Raptis Imperial collage**, **A. Mallios**

ASIDE

Problems caused by the continuum:

- singularities: caused by the geometric point like character of events
- non-normalizable infinities: caused by the fact that one can pack an infinite amount of pint-like events into a finite space.
- additional structure: the space-time manifold M needs extra structures put on top of it to make sense of the dynamics of General Relativity. those extra structures are: differential and Lorenzian metric.
- Space-time manifold topology seems to be a fixed rigid structure not partaking to any quantum dynamical fluctuation.