Topos Theory in the Foundations of Physics Talk 2

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"A theory is something nobody believes, except the person who made it. An experiment is something everybody believes, except the person who made it."

(Unknown)

Pure states and truth objects

In classical theory, a pure state is nothing but a point of state space.

Since the spectral presheaf $\underline{\Sigma}$ has no points, we must use another description for (pure) states, namely by certain elements of $P(P\underline{\Sigma})$. (In classical theory, both descriptions agree.)

Let ψ be a unit vector in Hilbert space. For each $V \in \mathcal{V}(\mathcal{H})$, we define

$$\begin{aligned} \mathbb{T}^{\psi}(V) &:= \{S \subseteq \underline{\Sigma}_V \mid \langle\psi|\widehat{P}_S|\psi\rangle = 1\} \\ &= \{S \subseteq \underline{\Sigma}_V \mid \widehat{P}_S \geq \delta(\widehat{P}_{\psi})_V\}. \end{aligned}$$

We call $\mathbb{T}^{\psi} = (\mathbb{T}^{\psi}(V))_{V \in \mathcal{V}(\mathcal{H})}$ the **truth object** corresponding to ψ.

The subobject classifier in $\mathbf{Set}^{\mathcal{V}(\mathcal{H})^{op}}$

The subobject classifier $\underline{\Omega}$ in a topos of presheaves is the presheaf of **sieves**.

A sieve in a poset like $\mathcal{V}(\mathcal{H})$ is particularly simple: let $V \in \mathcal{V}(\mathcal{H})$. A sieve α on V is a collection of subalgebras $V' \subseteq V$ such that, whenever $V' \in \alpha$ and $V'' \subset V'$, then $V'' \in \alpha$ (so α is a downward closed set).

The **maximal sieve** on V is $\downarrow V = \{V' \in \mathcal{V}(\mathcal{H}) \mid V' \subseteq V\}$.

A truth value is a global section of the presheaf $\underline{\Omega}$.

The global section consisting entirely of maximal sieves is interpreted as 'totally true', the global section consisting of empty sieves as 'totally false'.

Truth values from truth objects

We saw that subobjects of $\underline{\Sigma}$ represent propositions about the physical system under consideration, and that states are represented by truth objects.

Let $\underline{S} \in \mathrm{Sub}\,\underline{\Sigma}$ be such a subobject, and let $\mathbb{T}^{\psi}$ be a truth object.

Let

$$\nu(\ulcorner \underline{S} \urcorner \in \mathbb{T}^{\psi})_V := \{V' \subseteq V \mid \underline{S}(V') \in \mathbb{T}^{\psi}(V')\}.$$

One can show that this is a sieve on V. Moreover, for varying V, these sieves form a global section

$$\nu(\ulcorner \underline{S} \urcorner \in \mathbb{T}^{\psi}) \in \Gamma\underline{\Omega}.$$

This is the truth value of the proposition represented by $\underline{S}$, given by the truth object $\mathbb{T}^{\psi}$.

Natural transformations from operators

We want to represent physical quantities as natural transformations from the spectral presheaf to some presheaf related to the real numbers. Whenever $V' \subseteq V$, this will give a commutative diagram

$$\begin{array}{ccc} \underline{\Sigma}_V & \xrightarrow{\underline{\Sigma}(i_{V'V})} & \underline{\Sigma}'_V \\ \Big\downarrow{\scriptstyle \breve{\delta}(\widehat{A})_V} & & \Big\downarrow{\scriptstyle \breve{\delta}(\widehat{A})_{V'}} \\ \underline{\mathcal{R}}(V) & \xrightarrow[\rho(i_{V'V})]{} & \underline{\mathcal{R}}(V') \end{array}$$

At each stage V, the mapping

$$\breve{\delta}(\widehat{A})_V : \underline{\Sigma}_V \longmapsto \underline{\mathcal{R}}(V)$$

will be an evaluation, sending $\lambda_V \in \underline{\Sigma}_V$ to a real number $\lambda_V(\widehat{A}_V)$.

We construct $\widehat{A}_V$ from a given $\widehat{A} \in \mathcal{B}(\mathcal{H})_{sa}$ by a certain approximation, generalising the daseinisation of projections.

Daseinisation of self-adjoint operators

Let $\widehat{A} \in \mathcal{B}(\mathcal{H})_{sa}$. From the spectral family $\widehat{E}^A = (\widehat{E}^A_\lambda)_{\lambda \in \mathbb{R}}$, we obtain a new spectral family in $\mathcal{P}(V)$ by defining

$$\forall \lambda \in \mathbb{R} : \widehat{E}_\lambda^{\delta(\widehat{A})_V} := \bigvee \{ \widehat{Q} \in \mathcal{P}(V) \mid \widehat{Q} \leq \widehat{E}^A_\lambda \}.$$

This gives a self-adjoint operator $\delta(\widehat{A})_V$, which is the smallest operator in V larger than $\widehat{A}$ in the so-called 'spectral order'.

Similarly, we can define

$$\forall \lambda \in \mathbb{R} : \widehat{E}_\lambda^{\delta^i(\widehat{A})_V} := \bigwedge \{ \widehat{Q} \in \mathcal{P}(V) \mid \widehat{Q} \geq \widehat{E}^A_\lambda \}.$$

The corresponding operator $\delta^i(\widehat{A})_V$ approximates $\widehat{A}$ from below in the spectral order.

Spectral order

Def. (Olson '71, de Groote '04): Let $\widehat{A}, \widehat{B} \in \mathcal{B}(\mathcal{H})_{sa}$ with spectral families $\widehat{E}^A, \widehat{E}^B$. The **spectral order** is defined by

$$\widehat{A} \leq_s \widehat{B} :\Longleftrightarrow \forall \lambda \in \mathbb{R} : \widehat{E}^A_\lambda \geq \widehat{E}^B_\lambda .$$

- The spectral order is a partial order on the self-adjoint operators in $\mathcal{B}(\mathcal{H})$.
- On projections, the spectral order $<_s$ and the usual order $<$ coincide.
- Equipped with the spectral order, $\mathcal{B}(\mathcal{H})_{sa}$ becomes a boundedly complete lattice.
- The spectral order is coarser than the usual order on self-adjoint operators, i.e. $\widehat{A} <_s \widehat{B} \Longrightarrow \widehat{A} < \widehat{B}$.
- If $\widehat{A}$ and $\widehat{B}$ commute, then $\widehat{A} <_s \widehat{B} \Longleftrightarrow \widehat{A} < \widehat{B}$.

The mapping

$$\begin{aligned} \delta_V : \mathcal{B}(\mathcal{H})_{sa} &\longrightarrow V_{sa} \\ \widehat{A} &\longmapsto \delta(\widehat{A})_V \end{aligned}$$

adapts $\widehat{A}$ to the context V. The mapping δ_V is non-linear. We have

- $\mathrm{sp}(\delta(\widehat{A})_V) \subseteq \mathrm{sp}(\widehat{A})$.
- If $\widehat{A} = \widehat{P}$ is a projection, then $\delta(\widehat{A})_V$ is a projection, too, namely $\delta(\widehat{A})_V = \bigwedge\{\widehat{Q} \in \mathcal{P}(V) \mid \widehat{Q} \geq \widehat{P}\}$.
- $\delta(\widehat{A} + \alpha\widehat{I})_V = \delta(\widehat{A})_V + \alpha\widehat{I}$.
- If $\alpha \geq 0$, then $\delta(\alpha\widehat{A})_V = \alpha\delta(\widehat{A})_V$.
- $\delta(\widehat{A})_V$ is not a function of $\widehat{A}$ in general.

Analogous properties hold for δ^i_V.

The presheaf of order-reversing functions

We saw that for all $V' \subseteq V$, we have $\delta(\widehat{A})_{V'} \geq \delta(\widehat{A})_V$.

We define

$$\begin{aligned} \breve{\delta}(\widehat{A})_V : \underline{\Sigma}_V &\longrightarrow \underline{\mathcal{R}}(V) \\ \lambda &\longmapsto \{\lambda|_{V'}(\delta(\widehat{A})_{V'}) \mid V' \subseteq V\}. \end{aligned}$$

This is an order-reversing, real-valued function on the set $\downarrow V = \{V' \in \mathcal{V}(\mathcal{H}) \mid V' \subseteq V\}$.

These functions form a presheaf (Jackson '06) which we denote by $\underline{\mathbb{R}^{\succeq}}$. The restriction is simply given by restriction of the order-reversing functions.

By construction, $\breve{\delta}(\widehat{A})$ is a natural transformation from $\underline{\Sigma}$ to $\underline{\mathbb{R}^{\succeq}}$.

The quantity-value object $\underline{\mathbb{R}^{\leftrightarrow}}$

The presheaf $\underline{\mathbb{R}^{\succeq}}$ is a candidate for the quantity-value object for quantum theory.

Clearly, there is a similar construction giving a presheaf $\underline{\mathbb{R}^{\preceq}}$ of order-preserving functions, and for each $\widehat{A} \in \mathcal{B}(\mathcal{H})_{sa}$, a natural transformation $\breve{\delta}^{i}(\widehat{A}) : \underline{\Sigma} \rightarrow \underline{\mathbb{R}^{\preceq}}$, constructed using inner daseinisation of $\widehat{A}$.

We can combine both presheaves into one (product) presheaf $\underline{\mathbb{R}^{\leftrightarrow}}$. We consider this presheaf as the quantity value-object for QT.

Subobjects from pullbacks

A proposition of the form "$A \in \Delta$" refers to the real numbers, since $\Delta \subset \mathbb{R}$. The real numbers lie *outside* the topos $\mathbf{Set}^{\mathcal{V}(\mathcal{H})^{op}}$ (resp. the formal language, see later).

Now that we have defined $\underline{\mathbb{R}^{\leftrightarrow}}$, we can construct subobjects of $\underline{\Sigma}$ by pullback: let $\underline{\Theta}$ be a subobject of $\underline{\mathbb{R}^{\leftrightarrow}}$, then $\breve{\delta}(\widehat{A})^{-1}(\underline{\Theta})$ is a subobject of $\underline{\Sigma}$.

In this way, we get a topos-internal construction of propositions that do not refer to the real numbers. The 'meaning' of such propositions must be discussed from 'within the topos'.

Summary (up to now)

We had in *classical* theory:

- Physical quantities are morphisms (i.e., functions)

$$\mathcal{P} \xrightarrow{f_A} \mathbb{R},$$

- Propositions correspond to elements of Sub($\mathcal{P}$),
- (Pure) states are certain morphisms

$$\mathrm{Sub}(\mathcal{P}) \xrightarrow{\psi_p} \{0,1\}.$$

Classical physics is a **realist** theory. Logical formulas involving propositions can be manipulated according to the rules of a deductive system. The logic is Boolean, given by the topos Set of sets and mappings.

Summary (up to now)

We saw that in the topos formulation of quantum theory:

- Physical quantities are certain morphisms

$$\underline{\Sigma} \xrightarrow{\breve{\delta}(\widehat{A})} \underline{\mathbb{R}^{\leftrightarrow}}.$$

- Propositions correspond to subobjects $\underline{S} \in \mathrm{Sub}\,\underline{\Sigma}$ of the spectral presheaf $\underline{\Sigma}$,
- Pure states are certain morphisms

$$\mathrm{Sub}\,\underline{\Sigma} \xrightarrow{\mathbb{T}^{\psi}} \Gamma\underline{\Omega},$$

There is a deductive system in the form of intuitionistic logic, given by the internal logic of the presheaf topos $\mathbf{Set}^{\mathcal{V}(\mathcal{H})^{op}}$. This gives a **neo-realist** form of quantum theory.

Formal languages

There is a very elegant way of describing what we are doing: viz to construct a theory of a physical system S is equivalent to finding a representation in a topos of a certain formal language, $\mathcal{L}(S)$, that is attached to S.

- The language $\mathcal{L}(S)$ will depend on the physical system S, but not on the theory type (classical, quantum, ...).
- The representation will depend on the theory type.
- We want to allow for a logic that is not Boolean, but still is a deductive system. We choose *intuitionistic* axioms for the language.

The language $\mathcal{L}(S)$

The language $\mathcal{L}(S)$ of a system S is *typed*. It includes:

- A symbol Σ: the linguistic precursor of the state object.
- A symbol $\mathcal{R}$: the linguistic precursor of the quantity-value object.
- A set, $F_{\mathcal{L}(S)}(\Sigma, \mathcal{R})$ of 'function symbols' $A : \Sigma \to \mathcal{R}$: the linguistic precursors of physical quantities.
- A symbol Ω: the linguistic precursor of the sub-object classifier.
- A symbol 1: the linguistic precursor of the terminal object.
- A 'set builder' $\{\tilde{x} \mid \omega\}$. This is a term of type PT, where $\tilde{x}$ is a variable of type T, and ω is a term of type Ω.

Representing the language $\mathcal{L}(S)$

The next step is to find a representation ϕ of the language $\mathcal{L}(S)$ in a suitable topos.

In a classical theory of the system S, the representation σ is:

- The topos $\tau_\sigma(S)$ is Set.
- Σ is represented by a symplectic manifold $\Sigma_{\sigma,S}$.
- $\mathcal{R}$ is represented by the real numbers $\mathbb{R}$; i.e., $\mathcal{R}_{\sigma,S} := \mathbb{R}$.
- The function symbols $A : \Sigma \to \mathcal{R}$ become ordinary functions $A_{\sigma,S} : \Sigma_{\sigma,S} \to \mathbb{R}$.
- The symbol Ω is represented by the set $\{0, 1\}$ of truth values.

The topos of quantum theory

- The key ingredient of normal quantum theory on which we focus is the intrinsic *contextuality* implied by the Kocken-Specker theorem.
- In standard theory, we can potentially assign 'actual values' only to members of a commuting set of operators.

  We think of such a set as a context or 'classical snapshot' of the system.
- This motivates considering the topos $\mathrm{Set}^{\mathcal{V}(\mathcal{H})^{op}}$ of presheaves over the category $\mathcal{V}(\mathcal{H})$ of abelian subalgebras of $\mathcal{B}(\mathcal{H})$.

- The state object that represents the symbol Σ is the spectral presheaf $\underline{\Sigma}$.

  1. For each abelian subalgebra V, we have that $\underline{\Sigma}(V)$ is the spectrum of V.
  2. The KS theorem is equivalent to the statement that $\underline{\Sigma}$ has no global elements.
  3. The spectral presheaf replaces the (non-existent) state space.
  4. A proposition represented by a projector $\hat{P}$ in QT is mapped to a subobject $\delta(\hat{P})$ of $\underline{\Sigma}$. We call this '*daseinisation*'.

- The quantity-value symbol $\mathcal{R}$ is represented by a presheaf $\underline{\mathbb{R}^{\leftrightarrow}}$. This is *not* the real-number object in the topos.
- Physical quantities are represented by arrows $\breve{\delta}(\widehat{A}) : \underline{\Sigma} \rightarrow \underline{\mathbb{R}^{\leftrightarrow}}$. They are constructed from the Gel'fand transforms of daseinised self-adjoint operators.

Results by Spitters, Heunen and Landsman

Recently, Spitters and Heunen realised that results by Mulvey and Banaschewski show that the spectral presheaf can be understood as the internally and constructively defined Gel'fand spectrum of an internal abelian C^*-algebra. Seen externally, from an ambient topos (like Set), this C^*-algebra can well be non-abelian.

Moreover, the spectral presheaf is an internal **locale**, which allows further geometric study, e.g. by constructing $\mathrm{Sh}(\underline{\Sigma})$.

They also showed that the quantity-value presheaf $\underline{\mathbb{R}^{\leftrightarrow}}$ can be understood using Scott's interval domains. These are much used in theoretical computer science. They generalise Dedekind cuts in the sense that the lower and upper cut do not have to come arbitrarily close.

These results may allow for generalisations to other topoi (with a natural number object).

Results by Spitters, Heunen and Landsman

Moreover, there is a constructive theory of measures and integration, which works best for AW^*-algebras, since one needs lots of projections. This brings their work even closer to our von Neumann algebra-based scheme. We assume that we can use these results to derive ordinary expectation values and the Born rule from the topos framework.

Landsman has started to work on the axiomatic aspects of the topos framework. Presumably, his 'principle of tovariance' is closely related to the construction of suitable formal languages and their representations.

In his view, **geometric logic** plays a central rôle. This is the fragment of logical structure that is preserved under **geometric morphisms**.

Open problems and goals

There are many interesting open questions. Some of the things we are working on are:

- Description of commutators within the topos $\mathbf{Set}^{\mathcal{V}(\mathcal{H})^{op}}$, time evolution.
- Topos formulation of uncertainty relations.
- Superposition of states.
- Composite systems and entanglement.
- Internal vs. external formulations.
- Space-time concepts.
- ...

Reference:

A. Döring, C. J. Isham, "A Topos Foundation for Theories of Physics I-IV", quant-ph/0703060, 62, 64 and 66