Topos Theory in the Foundations of Physics
Talk 1

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"Grundlagenforschung betreibe ich dann, wenn ich nicht weiß, was ich tue."

("Fundamental research is what I am pursuing when I don't know what I am doing.")

Wernher von Braun

Motivation

The search for a theory of **quantum gravity** or **quantum cosmology** is difficult not "only" for technical, but also for conceptual reasons. One question is:

- Is the continuum picture of spacetime adequate at small scales?

Presumably not, something dramatic is expected at the Planck scale.

We measure physical quantities using rulers and pointers, so rely on smooth space-time structure.

- Do physical quantities necessarily have real numbers as values?

There is no *fundamental* reason to believe that this is the case in quantum gravity.

Further questions concern the status of measurements:

- Since there is no external observer in QC and QG, what would a measurement mean?
- No measurements means no relative-frequency interpretation of probabilities.

In QG, the use of the usual **instrumentalist** interpretation of QT as well as the use of the **continuum** (basically, the use of real and complex numbers) are doubtful. But:

- Standard quantum theory uses Hilbert spaces, operators, expectation values, path integrals, symmetries etc.; all are built upon the 'mathematical continuum', the real or complex numbers.
- All major approaches to QG use more-or-less standard quantum theory.

This may be wrong.

Topos theory as a new mathematical framework

In these talks, I want to show that **topos theory** allows to formulate physical theories in a way that is

- 'neo-realist' in the sense that propositions "$A \in \Delta$" about a physical system have truth-values, independent of measurements, observers etc. and
- does not fundamentally depend on the continuum in the form of the real numbers.

Of course, a theory of quantum gravity is still a long way off. I will concentrate on ordinary (algebraic) quantum theory and eventually say something on possible generalisations.

Plan of the talks

Talk 1:

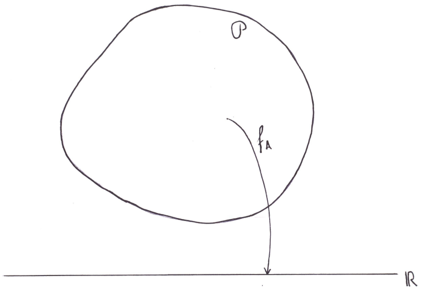
- Classical phase spaces and Boolean logic
- The Kochen-Specker theorem
- The spectral presheaf
- Definition of a topos and internal logic
- Daseinisation of projections

Talk 2:

- Pure states and truth objects
- Assignment of truth-values and 'neo-realism'
- Representation of self-adjoint operators as natural transformations
- Formal languages

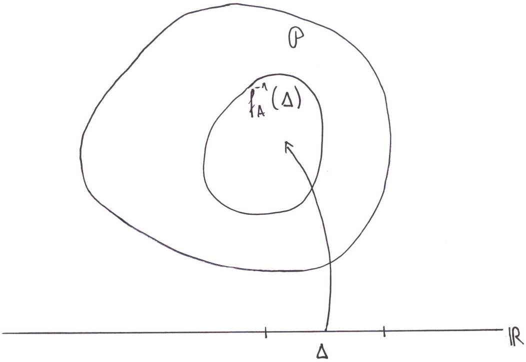
State spaces and Boolean logic

In classical physics: physical quantities/observables A are described by functions f_A on the state space $\mathcal{P}$, that is, mappings



Points of $\mathcal{P}$ are (pure) states.

One can also consider level sets, i.e., inverse images of elements $a \in \mathbb{R}$, or, more generally, inverse images of (Borel) subsets $\Delta \subseteq \mathbb{R}$:



Such a subset of the state space $\mathcal{P}$ corresponds to a **proposition** "$A \in \Delta$", that is, "the physical quantity A has a value lying in the set Δ".

Each point of the state space $\mathcal{P}$ either lies in $f_A^{-1}(\Delta)$ or not, i.e., in the state represented by the point the corresponding proposition is either true or false.

If we have two propositions, say "$A \in \Delta_1$" and "$B \in \Delta_2$" with corresponding subsets $f_A^{-1}(\Delta_1)$ and $f_B^{-1}(\Delta_2)$, then

- $f_A^{-1}(\Delta_1) \cap f_B^{-1}(\Delta_2)$ corresponds to the proposition "$A \in \Delta_1$ **and** $B \in \Delta_2$"
- $f_A^{-1}(\Delta_1) \cup f_B^{-1}(\Delta_2)$ corresponds to the proposition "$A \in \Delta_1$ **or** $B \in \Delta_2$"
- $\mathcal{P} \backslash f_A^{-1}(\Delta_1)$ corresponds to the **negation** "$A \notin \Delta_1$".

Classical physics is based on a **Boolean logical structure**, namely the Boolean lattice of subsets of state space.

Very schematically: in classical theory

- Physical quantities are morphisms (i.e., functions)

$$\mathcal{P} \xrightarrow{f_A} \mathbb{R},$$

- Propositions correspond to elements of Sub($\mathcal{P}$),
- States are certain morphisms

$$\mathrm{Sub}(\mathcal{P}) \xrightarrow{\psi_p} \{0,1\}.$$

Classical physics is a **realist** theory. Logical formulas involving propositions can be manipulated according to the rules of a **deductive system**.

The Kochen-Specker theorem

Problem: Is there a similar realist formulation of quantum theory?

More concretely, is there a 'state space' (space of hidden states) for QT such that physical quantities are real-valued functions (hidden variables) on this space? Self-adjoint operators $\widehat{A}$ should be embedded into the set of these functions.

Necessary condition for the existence of a space of hidden states: existence of valuation functions $v : \mathcal{R}_{sa} \rightarrow \mathbb{R}$ such that

(1) $v(\widehat{A}) \in \mathrm{sp}(\widehat{A})$ for all $\widehat{A} \in \mathcal{R}_{sa}$ (**spectrum rule**),

(2) For all bounded Borel functions $g : \mathbb{R} \rightarrow \mathbb{R}$, we have $v(g(\widehat{A})) = g(v(\widehat{A}))$ (**FUNC principle**).

The Kochen-Specker theorem

Kochen, Specker 1967: For $\mathcal{R} = \mathcal{B}(\mathcal{H})$, where $\dim \mathcal{H} \geq 3$, there are no valuation functions and hence no state space model of QT.

D 2005: This also holds for all unital von Neumann algebras $\mathcal{R}$ without summands of type I_1 and I_2.

In quantum theory, physical quantitities are represented by self-adjoint operators in $\mathcal{B}(\mathcal{H})$. The spectral theorem shows that to each proposition "$A \in \Delta$", there exists a projection operator $\widehat{E}[A \in \Delta] \in \mathcal{P}(\mathcal{H})$.

The KS theorem is equivalent to the fact that in quantum theory we cannot consistently assign "true" or "false" to all propositions at once (or 1 resp. 0 to the projections corresponding to the propositions).

Contexts or Weltanschauungen

- There is no model of QT in which all physical quantities have values at once. What becomes of realism?
- Not surprisingly, there is no problem for *abelian* algebras. The operators in an abelian C^*-algebra can be written as continuous functions (Gel'fand transforms) on the Gel'fand spectrum.
- Abelian subalgebras of $\mathcal{B}(\mathcal{H})$ are called **contexts**. They are like 'classical snapshots' of the quantum system.
- Some kind of *contextual* model of QT is needed (but with good control of the relations between contexts).
- Also needed: some novel mathematical model for propositions and their logic.

The context category

We consider the category $\mathcal{V}(\mathcal{H})$ of non-trivial unital abelian von Neumann subalgebras of $\mathcal{B}(\mathcal{H})$. This is a poset and is called the **context category**.

We use von Neumann algebras (rather than C^*-algebras) since

- They have enough projections; their projection lattices are complete.
- The spectral theorem holds for von Neumann algebras, which gives the connection between propositions "$A \in \Delta$"and projections.

We exclude the trivial algebra $\mathbb{C}\widehat{1}$. The minimal elements in $\mathcal{V}(\mathcal{H})$ are of the form $V_{\widehat{P}} := \{\widehat{P}, \widehat{1}\}'' = \mathbb{C}\widehat{P} + \mathbb{C}\widehat{1}$.

Gel'fand spectrum and Gel'fand transformation

Let V be an abelian von Neumann algebra. The **Gel'fand spectrum** $\underline{\Sigma}_V$ of V is the set of all pure states of V.

Equivalently, $\underline{\Sigma}_V$ is the set of multiplicative states and the set of algebra homomorphisms $\lambda : V \to \mathbb{C}$.

$\underline{\Sigma}_V$ is equipped with the weak*-topology and thus becomes a compact Hausdorff space. For an abelian von Neumann algebra, $\underline{\Sigma}_V$ is extremely disconnected.

The **Gel'fand transformation** is the mapping

$$\begin{aligned} V &\longrightarrow C(\underline{\Sigma}_V) \\ \widehat{A} &\longmapsto \overline{A}, \end{aligned}$$

where, for all $\lambda \in \underline{\Sigma}_V$, we have $\overline{A}(\lambda) := \lambda(\widehat{A})$.

The spectral presheaf

To each $V \in \mathcal{V}(\mathcal{H})$, we assign its Gel'fand spectrum $\underline{\Sigma}_V$.

If $V' \subseteq V$, we have a morphism $i_{V'V} : V' \to V$ in the context category $\mathcal{V}(\mathcal{H})$ and define

$$\begin{aligned} \underline{\Sigma}(i_{V'V}) : \underline{\Sigma}_V &\longrightarrow \underline{\Sigma}_{V'} \\ \lambda &\longmapsto \lambda|_{V'}. \end{aligned}$$

$\underline{\Sigma}$ is a contravariant functor from the context category $\mathcal{V}(\mathcal{H})$ to the category Set, i.e. a **presheaf over** $\mathcal{V}(\mathcal{H})$. Notation: $\underline{\Sigma} = (\underline{\Sigma}_V)_{V \in \mathcal{V}(\mathcal{H})} \in \mathbf{Set}^{\mathcal{V}(\mathcal{H})^{op}}$.

We regard the **spectral presheaf** $\underline{\Sigma}$ as a quantum analogue of state space.

Reformulation of the KS theorem

Does $\underline{\Sigma}$ have **global sections**? I.e., can we pick one element λ_V from each Gelfand spectrum $\underline{\Sigma}_V$, $V \in \mathcal{V}(\mathcal{H})$, such that, whenever $V' \subset V$, we have $\lambda_{V'} = \lambda_V|_{V'}$?

Isham, Butterfield '98: a global section would give a valuation function $v : \mathcal{B}(\mathcal{H})_{sa} \to \mathbb{R}$ and vice versa. So we have a reformulation of the Kochen-Specker theorem:

Thm.: The spectral presheaf $\underline{\Sigma}$ has no global sections.

Global sections as points

A global section - if it existed - would be a morphism

$$\underline{1} \longrightarrow \underline{\Sigma}$$

from the terminal object $\underline{1}$ of $\mathrm{Set}^{\mathcal{V}(\mathcal{H})^{op}}$ to $\underline{\Sigma}$.

In Set, the terminal object 1 is the one-element set. If S is some set, then a morphism $1 \to S$ picks one element of S, so each morphism $1 \to S$ can be interpreted as a **point of** S.

The fact that there are no morphisms $\underline{1} \to \underline{\Sigma}$ means that $\underline{\Sigma}$ *has no points*.

Ordinary Quantum Logic

Birkhoff, von Neumann 1936: lattice $L(\mathcal{H})$ of closed subspaces of Hilbert space $\mathcal{H}$ describes the logic of quantum systems.

- At first sight, this is similar to a classical propositional calculus with the Hilbert space $\mathcal{H}$ taking the rôle of the quantum state space analogue.
- Severe interpretational problem: if $\dim \mathcal{H} > 1$, then $L(\mathcal{H})$ is **non-distributive**. Example: the "quantum breakfast"

$$E \wedge (B \vee S) \neq (E \wedge B) \vee (E \wedge S).$$

- There are many further developments in **quantum logic**, but these are somewhat detached from physics.
- In particular, a viable deductive system is lacking.

The topos of presheaves

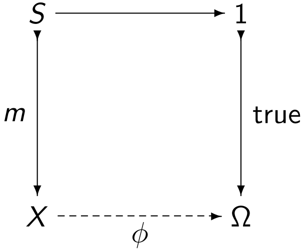
We now want to use the fact that the presheaf category $\mathrm{Set}^{\mathcal{V}(\mathcal{H})^{op}}$ is a **topos** with an **internal logic**.

A topos is a category that is similar to Set. Specifically, in Set, we have the operations of disjoint union, cartesian product and exponentiation (taking S^T, the set of all functions from T to S).

Correspondingly, in a topos, we have (finite) **colimits**, **limits** and **exponentials**.

Moreover, each topos has a so-called **subobject classifier**. This is a special object Ω in the topos that generalises the set $\{0, 1\}$ of truth-values in the category Set. (Of course, Set is a topos, too.)

Def.: In a category $\mathcal{C}$ with finite limits, a **subobject classifier** is a monic, true : $1 \to \Omega$, such that to every monic $S \to X$ in $\mathcal{C}$ there is a unique morphism ϕ which, with the given monic, forms a pullback diagram:



Topos logic

The subobject classifier endows a topos with an *internal logic*, which is of **intuitionistic** type. The important result that we use is:

Thm.: The set $\mathrm{Sub}\,\underline{\Sigma}$ of subobjects of the spectral presheaf forms a **Heyting algebra**.

A Heyting algebra is a distributive, pseudo-complemented lattice in which the law of excluded middle need not hold. This means that we can get an ordinary breakfast ;-)

$$E \wedge (B \vee S) = (E \wedge B) \vee (E \wedge S),$$

but we have

$$E \vee \neg E \leq 1,$$

which is equivalent to

$$\neg\neg E \geq E.$$

Daseinisation of projections

Let $\widehat{P} = \widehat{E}[A \in \Delta] \in \mathcal{P}(\mathcal{H})$ be the projection corresponding to the proposition "$A \in \Delta$".

In order to relate $\widehat{P}$ to *all* the contexts $V \in \mathcal{V}(\mathcal{H})$, we define a mapping

$$\begin{aligned} \delta : \mathcal{P}(\mathcal{H}) &\longrightarrow (\mathcal{P}(V))_{V \in \mathcal{V}(\mathcal{H})} \\ \widehat{P} &\longmapsto (\delta(\widehat{P})_V)_{V \in \mathcal{V}(\mathcal{H})}, \end{aligned}$$

where

$$\delta(\widehat{P})_V := \bigwedge \{\widehat{Q} \in \mathcal{P}(V) \mid \widehat{Q} \geq \widehat{P}\}.$$

That is, we approximate $\widehat{P}$ from above in each context. If $\widehat{P} \in V$, then $\delta(\widehat{P})_V = \widehat{P}$.

Daseinisation of projections

Each projection $\delta(\widehat{P})_V \in \mathcal{P}(V)$ defines a subset of the Gel'fand spectrum $\underline{\Sigma}_V$ by

$$\delta(\widehat{P})_V \longmapsto \{\lambda \in \underline{\Sigma}_V \mid \lambda(\delta(\widehat{P})_V) = 1\}.$$

In fact, for each V, we get a clopen subset of $\underline{\Sigma}_V$. One can easily show that these subsets fit together to form a subobject of $\underline{\Sigma}$. We obtain a mapping

$$\delta : \mathcal{P}(\mathcal{H}) \longrightarrow \mathrm{Sub}\,\underline{\Sigma}$$

from the projection lattice into the subobjects of $\underline{\Sigma}$, which we call the **daseinisation** of P.

Daseinisation is injective and order-preserving.

Daseinisation sends projections to subobjects of $\underline{\Sigma}$,

$$\delta : \mathcal{P}(\mathcal{H}) \longrightarrow \mathrm{Sub}\,\underline{\Sigma},$$

so we have mapped propositions about a quantum system to a *distributive* lattice in a contextual manner.

One can show that

$$\delta(\widehat{P} \vee \widehat{Q}) = \delta(\widehat{P}) \vee \delta(\widehat{Q}),$$

but

$$\delta(\widehat{P} \wedge \widehat{Q}) \leq \delta(\widehat{P}) \wedge \delta(\widehat{Q}).$$

Not every subobject of $\underline{\Sigma}$ comes from a projection. For example, the subobject $\delta(\widehat{P}) \wedge \delta(\widehat{Q})$ is not of the form $\delta(\widehat{R})$ for any projection $\widehat{R}$ in general.

A baby version of the uncertainty relations

We want to interpret the formula $\delta(\widehat{P} \wedge \widehat{Q}) \leq \delta(\widehat{P}) \wedge \delta(\widehat{Q})$. Let $\widehat{P}$ be the projection representing "$p \in \Delta$", and let $\widehat{Q}$ represent "$q \in \Gamma$".

In $\mathrm{Sub}\,\underline{\Sigma}$, the proposition "$p \in \Delta$ and $q \in \Gamma$" is represented by $\delta(\widehat{P}) \wedge \delta(\widehat{Q}) = (\delta(\widehat{P})_V \wedge \delta(\widehat{Q})_V)_{V \in \mathcal{V}(\mathcal{H})}$.

There is no context $V \in \mathcal{V}(\mathcal{H})$ that contains both $\widehat{P}$ and $\widehat{Q}$, so, in each V, at least one of the projections $\widehat{P}, \widehat{Q}$ must be adapted by actually making it larger. Going to a larger projection means making the corresponding proposition coarser.

This means that when representing a proposition "$p \in \Delta$ and $q \in \Gamma$", we have a built-in coarse-graining which, intuitively, does not allow both Δ and Γ to become arbitrarily small at once (that is, in the same context or classical perspective V).

Pure states and truth objects

In classical theory, a pure state is nothing but a point of state space.

Since the spectral presheaf $\underline{\Sigma}$ has no points, we must use another description for (pure) states, namely by certain elements of $P(P\underline{\Sigma})$. (In classical theory, both descriptions agree.)

Let ψ be a unit vector in Hilbert space. For each $V \in \mathcal{V}(\mathcal{H})$, we define

$$\begin{aligned} \mathbb{T}^{\psi}(V) &:= \{S \subseteq \underline{\Sigma}_V \mid \langle\psi|\widehat{P}_S|\psi\rangle = 1\} \\ &= \{S \subseteq \underline{\Sigma}_V \mid \widehat{P}_S \geq \delta(\widehat{P}_\psi)_V\}. \end{aligned}$$

We call $\mathbb{T}^{\psi} = (\mathbb{T}^{\psi}(V))_{V\in\mathcal{V}(\mathcal{H})}$ the **truth object** corresponding to ψ.

The subobject classifier in $\mathbf{Set}^{\mathcal{V}(\mathcal{H})^{op}}$

The subobject classifier $\underline{\Omega}$ in a topos of presheaves is the presheaf of **sieves**.

A sieve in a poset like $\mathcal{V}(\mathcal{H})$ is particularly simple: let $V \in \mathcal{V}(\mathcal{H})$. A sieve α on V is a collection of subalgebras $V' \subseteq V$ such that, whenever $V' \in \alpha$ and $V'' \subset V'$, then $V'' \in \alpha$ (so α is a downward closed set).

The **maximal sieve** on V is $\downarrow V = \{V' \in \mathcal{V}(\mathcal{H}) \mid V' \subseteq V\}$.

A truth value is a global section of the presheaf $\underline{\Omega}$.

The global section consisting entirely of maximal sieves is interpreted as 'totally true', the global section consisting of empty sieves as 'totally false'.

Truth values from truth objects

We saw that subobjects of $\underline{\Sigma}$ represent propositions about the physical system under consideration, and that states are represented by truth objects.

Let $\underline{S} \in \operatorname{Sub} \underline{\Sigma}$ be such a subobject, and let $\mathbb{T}^{\psi}$ be a truth object.

Let

$$\nu(\ulcorner \underline{S} \urcorner \in \mathbb{T}^{\psi})_V := \{V' \subseteq V \mid \underline{S}(V') \in \mathbb{T}^{\psi}(V')\}.$$

One can show that this is a sieve on V. Moreover, for varying V, these sieves form a global section

$$\nu(\ulcorner \underline{S} \urcorner \in \mathbb{T}^{\psi}) \in \Gamma \underline{\Omega}.$$

This is the truth value of the proposition represented by $\underline{S}$, given by the truth object $\mathbb{T}^{\psi}$.

References

A. Döring, C. J. Isham, “A Topos Foundation for Theories of Physics I-IV”, quant-ph/0703060, 62, 64 and 66