A brief summary of State-Vector Reduction the language of Topos Theory

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Chapter 1

Summary of State-Vector Reduction paper

In this paper I will give a summary of the description of State Vector reduction using Topos Theory put forward by Isham in [1]. I will first briefly explain some mathematical concepts regarding Categories and Topoi. For a more detailed and complete explanation of the subjects of Topos and Category Theory the readers should refer to Topos-Physics and reference therein. I would like to point out that in this short summary I will not report all of the concepts delineated in [1], but I will just give a general overview of the main results. I will, instead, leave it to the curious reader to explore in her/his own time the details of the paper [1].

1.1 Preliminary definitions

- **State-Vector reduction**

  Given a state ψ, the process of State Vector reduction is the process whereby, after a series of "ideal" measurements, represented by the projection operators $\hat{P}_1 \cdots \hat{P}_n$, the system is "transformed" to

  $$|\psi\rangle \rightarrow \hat{P}_n \hat{P}_{n-1} \cdots \hat{P}_1 |\psi\rangle$$

- **M-set**

  Let the triple $M = (\mathcal{M} * e)$ be a monoid [1], then for all $m \in \mathcal{M}$ there exists a map $\lambda_m : \mathcal{M} \rightarrow \mathcal{M}$ (which denotes left multiplication be m), such that $\lambda_m(n) = m * n$. It is possible to generalize this definition for any set X on which $\mathcal{M}$ acts on the left, obtaining the following:

  $\lambda_m : X \rightarrow X$ such that

  1) $\lambda_e = id_x$

  2) $\lambda_m o \lambda_p = \lambda_{m*p}$

  Therefore, the collection of all λ_m for all $m \in \mathcal{M}$ defines the complete action of $\mathcal{M}$ on X which can be denoted by the general map $\lambda : \mathcal{M} \times X \rightarrow \mathcal{M}$. We can now identify the pair (X, λ) with an (left) M-set

- **Equivariant maps**

  Given two M-sets (X, λ) and (Y, μ) we define an equivariant map to be a

[1] A monoid M is a triple $M = (\mathcal{M} * e)$ where a) $\mathcal{M}$ is a set b) $*$ is a binary operation on $\mathcal{M}$ i.e. $* : \mathcal{M} \rightarrow \mathcal{M}$ c) $e \in \mathcal{M}$ is the identity element such that $e * x = x * e = x$

map $f : (X, \lambda) \rightarrow (Y, \mu)$ which makes the following diagram commute

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \lambda_m \downarrow & & \downarrow \mu_m \\ X & \xrightarrow[f]{} & Y \end{array}$$

i.e $\mu_m o f = f o \lambda_m$

- **Left ideal**

  Given a monoid M, a subset $B \subseteq M$ is said to be a left ideal of M provided it is closed under left multiplication i.e $m * b \in B$ whenever $b \in B$ and $m \in M$. Schematically the condition of left ideal can be written as follows: $B \subseteq M$ is a left ideal iff $mB := \{mb \in M | b \in B\} \subset B$ for all $m \in M$. (Note that $\emptyset$ and M are themselves left ideals)

- **BM Category**

  A BM category is a category which has the following:

  1) objects are left M-sets (defined above)

  2) arrows are equivariant maps (defined above)

- **Subobject classifier in BM**

  The definition of a Subobject classifier (See Subobject classifier for definition) requires the existence of two elements:

  1) a truth object Ω

  2) and a truth arrow $T : 1 \rightarrow \Omega$.

  We would now like to know what these elements are with respect to the category BM.

  1) In BM the truth object is identified with the "pair" $\Omega(L_M, w)$ where,

L_M is the set of left ideal on M (note that $\emptyset$ and $\mathcal{M}$ belong to L_M) and $w : M \times L_M \to L_M$ is such that $w(m, B) = \{m^{'} \in M | m^{'}m \in B\} = w_m(B)$ is a left ideal.

It is easy to show that composition is defined as follows $w_m o w_n = w_{mn}$.

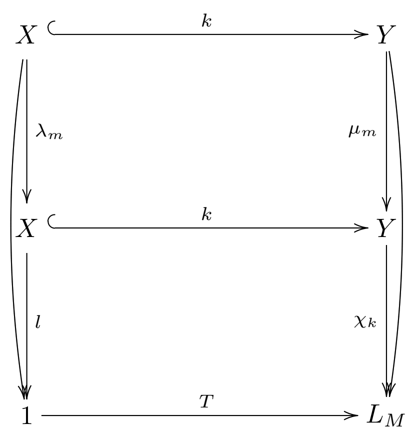
In fact we have

$$
\begin{aligned}
(w_n o w_m)B &= w_n(w_m(B)) \\
&= w_m o\{m^{'} \in M | m^{'}m \in B\} \\
&= \{n^{'}m^{'} \in M | n^{'}m^{'}nm \in B\} \\
&= w_{nm}
\end{aligned}
$$

2) In the category BM the arrow True $T : 1 \to \Omega$ is a function $T : \{0\} \to L_M$ such that, T(0)=M is the largest left ideal on M.

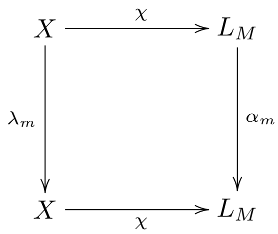
Given two elements $X, Y \in BM$ and an inclusion map $k : X \to Y$ (which is equivariant), the workings of the T arrow can be described by the following diagram:

Diagram 1.1



The map χ, which appears in the above diagram, is called a characteristic function and is defined as follows: for a given m, χ is an M-equivariant map, i.e.

Diagram 1.2



commutes.

If we restrict the definition χ to a particular inclusion map (our case k), then χ is identified with the map $\chi_k : (Y, M) \to \Omega$ such that, for a particular $m \in M$, we have $\chi_k : Y \to L_M$. We now want to show that the diagram 1.1 commutes, this can be easily done since each of the 3 component diagrams (2 small squares and the big rectangle) commute.

The fact that diagram 1.1 commutes implies that,for all $y \in Y$, the action of the map χ_k can be defined as follows.

$$\chi_k(y) = \{m|\mu_m(y) \in X\} \tag{1.1}$$

Since we know that L_M is the set of left ideals on M, in order to check whether the definition of $\chi_k(y)$ by 1.1 is correct, we need to show that $\chi_k(y)$ is a left ideal.

Proof 1.1 *We know that* $X \in BM$*, therefore* $\forall m \in M$ *and* $x \in X$*,* $m * x \in X$*. From definition of* $\chi_k(y) = \{m|\mu_m(y) \in X\}$ *it is clear that* $\chi_k(y) \subseteq M$ *since* $\chi_k(y)$ *contains the subset of elements in M such that the condition* $\mu_m(y) \in X$ *is satisfied. So all we need to show is that*

$m^{'}\mu_m(y) \in X$ for all $m^{'} \in M$. Call $\mu_m(y) = x$ then, since $X \in BM$, it follows that $m^{'} * x \in X$ $\forall m^{'} \in M$.

Conversely if we were given a general equivariant map $\chi : Y \to L_m$ then, $X^{\chi} \subset Y := \{y \in Y | \chi(y) = M = 1\}$ is an M-invariant subset of Y for $Y \in BM$.

Proof 1.2 *Recall that M is itself a left ideal therefore, since χ is an equivariant map (recall diagram 1.2) we have the following equality*

$$\chi o \lambda_m(x) = \chi(m * x) = \alpha_m o \chi(x)$$

We now want to show that $\forall m \in M$ $m * x \in X$ for $x \in X$, i.e. we want to show that $\chi(m * x) = M$. Since χ is an equivariant map we get

$$\begin{aligned} \chi o \lambda_m(x) &= \chi(m * x) \\ &= \alpha_m o \chi(x) \\ &= \alpha_m M \\ &= m * M \\ &= M \end{aligned}$$

- **Topos BM**

  The category BM defined above is actually a Topos (see Topos), i.e. it can be shown that BM has the following objects:

  1. BM has an initial (0) and a terminal (1) object
  2. BM has pullbacks
  3. BM has pushouts

4. BM has exponentiation, i.e. BM is such that for every pair of objects X and Y in BM exists the map Y^X
5. T has a subobject classifier

1.2 Topos interpretation of State-Vector reduction and Truth Values

In order to understand the interpretation of State Vector reduction using Topos Theory put forward by Isham, we first need to understand how truth values for certain propositions can be defined in terms of Topos Theory. The connection between the two will be explained later on in this summary. For a more in depth discussion on truth values defined in terms of Topos Theory, the reader should refer to Topos and Logic.

From Topos Theory we know that truth values are identified with elements of the subobject classifier. In the Topos of M-sets elements of the subobject classifier are **left ideals**, therefore we, somehow, have to try to interpret equation 1.1 as representing the truth value of a proposition. In his paper, Isham suggests an interpretation of equation 1.1 as an indication of how close is an element y, to belong or not, to a certain set (in this case X). Pictorially speaking, the right side of equation 1.1 should be seen as an indication/quantification of the amount of different ways(=elements of M) in which an element y can be made to belong to a subset X, such that, if an element y actually belongs to X, it will continue doing so after any element of M is applied to it. For example given $X := \{x \in X|y = \text{human female, short black hair}\}$ we then consider two elements $y_1 = \{y_1 \in Y | y_1 = \text{male, blond, long hair, brown eyes}\}$ and $y_2 = \{y_2 \in Y | y_2 = \text{female, brown eyes, blond,black hair}\}$ in the set $Y := \{y \in Y | y = human\}$ and . Now if the monoid M were formed by $M := \{\text{dye hair, cut hair and put blue contact lenses, cut hair and put glasses}\}$, then any but one of the elements of M applied to y_2 would send y_2 in X, while no element of M would send y_1 in X. Therefore we would say that y_2 is nearly in X while y_1 is not at all in X. This is obviously a very simple example, but it gives the general idea behind the concept of truth values using Topos Theory put forward by Isham in [1], and, more important it conveys the idea that the truthfulness of a proposition (in the example above ($y_2 \in X$)) is directly proportionate to the amount of elements of a given monoid M which "send" y_2 into. In fact, for a given Topos, Isham identifies the set of truth values for the proposition "y is an element of X" as follows

$$[y \in X]_{BM} := \{m \in M | my \in X\} \tag{1.2}$$

where the right hand side is a left ideal in the monoid M, as was proved above (see equation 1.1 and proof). If y is actually in X then, the right hand side of equation 1.2 will be M itself.

Furthermore it can be shown that 1.2 can be generalized for any subset $K \subset Y$ as follows:

$$[y \in K]_{BM} := \{m \in M | my \in mK\} \tag{1.3}$$

where $mK = K_m := \{my \in K_m | y \in Y\}$

The importance of defining truth values in terms of a Topos of M-sets, as in equation 1.3, is the fact that it immediately suggests a way of describing the process of state-vector reduction. In fact, as previously stated, the right hand side of equation 1.3 is to be interpreted as the amount of possible "changes" that can be made to a certain element with it still belonging to a given subset. If we now tried to find, in Quantum Mechanics, an analogue of this procedure the first thing that comes to mind is the process of State Vector reduction (see 1.1). In fact, if we had a certain subset $\mathcal{H}_{\hat{P}}$, of $\mathcal{H}$ and some vector $|\psi\rangle \notin \mathcal{H}_{\hat{P}}$, we could apply a series of "transformations" (finite products of projection operators) represented by projection operators, such that $\hat{P}_n \cdots \hat{P}_1|\psi\rangle \in \mathcal{H}_{\hat{P}}$. In his paper Isham used this idea to define a quantum analogue of equation 1.3. The Topos (BM), which he chose for this task is, the Topos $BPrP(\mathcal{H})$, where $PrP(\mathcal{H})$ is the set formed by all finite products of projection operators, and the elements of $BPrP(\mathcal{H})$ are subsets of the Hilbert space on which these finite products act on the left. A more in depth explanation of the Topos $BPrP(\mathcal{H})$ and its derivation will be given in the next section, now I will just report how truth values of Quantum Propositions (quantum analogue of equation 1.3) are defined within $BPrP(\mathcal{H})$. Namely we have:

$$[|\psi\rangle \in \mathcal{H}_{\hat{P}}]_{BPrP(\mathcal{H})} = \{\hat{P}_n \cdots \hat{P}_1 | \hat{P}_n \cdots \hat{P}_1|\psi\rangle \in \hat{P}_n \cdots \hat{P}_1|\psi\rangle \in \mathcal{H}_{\hat{P}}\} \tag{1.4}$$

where the same set of "transformations" is applied to both the vector $|\psi\rangle$ and to the Hilbert space $\mathcal{H}_{\hat{P}}$. We could also write equation 1.4 in terms of rays and projective limit:

$$[[|\psi\rangle] \in P\mathcal{H}_{\hat{P}}]_{BPrP(\mathcal{H})} = \{\hat{P}_n \cdots \hat{P}_1 | [\hat{P}_n \cdots \hat{P}_1|\psi\rangle] \in l_{\hat{P}_n \cdots \hat{P}_1} P\mathcal{H}_{\hat{P}}\} \tag{1.5}$$

As can be seen, truthfulness of a proposition is directly proportioned to the amount of elements contained/allowed in the right hand side of 1.4 or 1.5.

It is now appropriate to give some explanations on equations 1.4 and 1.5. Since these equations are related to the process of State Vector reduction, I will now describe how this process is interpreted in the language of Topos theory.

1.2.1 State-Vector Reduction in the language of Topos Theory

The first step towards the description of the process of State-Vector reduction using Topos Theory, is to choose a suitable monoid M in terms of which to define the Topos BM. One natural choice would be to identify the set of all bounded linear operators L($\mathcal{H}$) on $\mathcal{H}$ with a monoid. This identification is correct since each element of the triple $(L(\mathcal{H}), *, e)$ is well defined. In fact the $*$ operation can be identified with operator product, while the identity element e can be associated with the unit operator $\hat{1}$. Since we would like to work in the Topos BM, we need to define a set on which the monoid L($\mathcal{H}$) acts on the left. For this set we have two choices:

1. we could either have the set $\mathcal{H}$
2. or we could have the set of projective Hilbert spaces $P\mathcal{H}$.[2]

For choice number 1 we can define the action of L($\mathcal{H}$) on $\mathcal{H}$ as follows:

$$l_{\hat{A}}(|\psi\rangle) := \hat{A}|\psi\rangle$$

$\forall\ \hat{A} \in L(\mathcal{H})$ and $|\psi\rangle \in \mathcal{H}$.

For choice number 2 instead we would have the following action:

$$l_{\hat{A}}([|\psi\rangle]) := [\hat{A}|\psi\rangle]$$

[2] The projective Hilbert space of a complex Hilbert space $\mathcal{H}$ is the set of equivalence classes of vectors V in $\mathcal{H}$ with $V \neq 0$. The equivalence relation is given by the following $V \sim W$ iff $V = \lambda W$, where λ is a scalar. The equivalence classes with respect to $\sim$ are called the projective rays. The definition of projective Hilbert space implies that the states ψ and $\lambda\psi$ represent the same physical state.

where $[\hat{A}|\psi\rangle]$ denotes the projective ray that passes through $\hat{A}|\psi\rangle$[3] At this point it is worth noting that the action of L($\mathcal{H}$) can be extended to any close linear subspaces of both $\mathcal{H}$ and $P\mathcal{H}$ (I will not go into the details of the above statement, the reader should refer to [1]).

So, in Quantum Mechanics, we have ,now, found a possible candidate for a monoid i.e. L($\mathcal{H}$). We know that the set of all M-sets (sets on which the monoid M acts on the left) form a topos, since L($\mathcal{H}$) acts on $\mathcal{H}$, we could identify the Topos B$L(\mathcal{H})$ with the set of all Hilbert spaces. The only problem with this construction is that L($\mathcal{H}$) has no real interesting physical meaning, therefore, we need to choose some subset of $L(\mathcal{H})$ to which can be given a physical interpretation. Isham, in his paper, chooses the subset Pr$P(\mathcal{H}) \subset L(\mathcal{H})$ of all finite products of projection operators. The action of Pr$P(\mathcal{H})$ on $\mathcal{H}$ would then be the following:

$\hat{P}_n \cdots \hat{P}_1|\psi\rangle = |\omega\rangle$

We can now give a well defined mathematical context to the previously just stated equations 1.4, and 1.5. Namely, within the context of the Topos of Pr$P(\mathcal{H})$-sets, the elements of the subobject classifier are identified with the right hand side of equations 1.4, or 1.5 which are left ideals, therefore they represents truth-values in the Topos Pr$P(\mathcal{H})$-sets. Moreover, since truth values can be interpreted as a set of possible "transformations", the right hand side of equations 1.4, and 1.5 is ulteriorly interpreted as exemplifying the process of State Vector reduction. There are, however, two problems concerning the identification of $|\psi\rangle \rightarrow \hat{P}_n \cdots \hat{P}_1|\psi\rangle$ with the process of State Vector reduction,

[3]Note that if $\hat{A}|\psi\rangle = 0$, then $[\hat{A}|\psi\rangle] = [0]$ where $[0]$ is such that $l_{\hat{A}}([0]) = [0]$. Therefore the action of L($\mathcal{H}$) should really be on $P\mathcal{H} \cup [0]$, where $[0]$ should be identified with the accumulation point of the action.

namely

1. Non uniqueness of product of projections: more than one collection of projection operators have the same product.

2. Normalisation issue: State Vector reduction should really be of the form

$$|\psi\rangle \rightarrow \frac{\hat{P}_n \cdots \hat{P}_1|\psi\rangle}{||\hat{P}_n \cdots \hat{P}_1|\psi\rangle||} \tag{1.6}$$

therefore we can not allow $||\hat{P}_n \cdots \hat{P}_1|\psi\rangle|| = 0$

1. Problem number one can be solved by choosing, instead of a monoid formed by products of projection operators, a monoid formed by strings of projection operators, i.e. $SP(\mathcal{H})$, whose elements are $P := \{\hat{P}_n, \hat{P}_{n-1}, \cdots, \hat{P}_1\}$. Within this monoid the product is identified with the concatenation of strings, while the unit element is the unit operator. Given this monoid, the action of a string P on a state $|\psi\rangle$ would be identified (counterfactually) as the subsequent application of the individual projector operators which make up the string, i.e. $\hat{P}|\psi\rangle = \hat{P}_n\hat{P}_{n-1} \cdots \hat{P}_1|\psi\rangle$, where $\hat{P} = \hat{P}_n\hat{P}_{n-1} \cdots \hat{P}_1$ is called the reduction of $P := \{\hat{P}_n, \hat{P}_{n-1}, \cdots, \hat{P}_1\}$. It is easy to see that if we adopted this new monoid, problem 1) would disappear, since we are now dealing with strings of projectors rather than products, and strings are uniquely defined. It is only counterfactually that we define the action of a string on a State Vector as a succession of application of the individual projector operators which make up the string.
   In this context, equation 1.5 becomes

$$[[|\psi\rangle] \in P\mathcal{H}_{\hat{P}}]_{BSP(\mathcal{H})} = \{Q \in SP(\mathcal{H}) | [\hat{Q}|\psi\rangle] \in l_{\hat{Q}} P\mathcal{H}_{\hat{P}}\} \tag{1.7}$$

2. Regarding the normalisation issue, Isham, in his paper [1], proposed two

methods to solve Normalisation issue. I will hereby shortly report both of them

- The first method involves the introduction of a new type of category whose creation is based on the following reasoning: we want the monoid $SP(\mathcal{H})$ so that we can deal with the "uniqueness" problem, but we also require strings $P \in SP(\mathcal{H})$ to be such that the product of the individual projectors composing the string is different form zero. So, what we are really interested in, is the monoid $SP(\mathcal{H})_0$ whose elements are such that the property $\hat{P} \neq 0$ holds. The caveat with this monoid is that it is not closed under product composition, i.e. given $P \neq 0$ and $Q \neq 0$ it is not necessarily the case that $PQ \neq 0$. This feature makes $SP(\mathcal{H})_0$ a partial monoid. A way of taking into account this feature is by defining a new category χ, such that
  - Objects Ξ are identified with collections of non zero vectors in $\mathcal{H}$ such that if $|\psi\rangle \in \Xi$, then $\lambda|\psi\rangle \in \Xi$ for all $\lambda \in \mathbb{C}_*$ [4]
  - Arrows are identified with $Hom(\Xi_1, \Xi_2) : \{P \in SP(\mathcal{H})_0 | \forall |\psi\rangle \in \Xi_1, \hat{P}|\psi\rangle \in \Xi_2\}$ and the composition of arrows is identified with the concatenation of strings.

  As we can see, the elements of the monoind $SP(\mathcal{H})$ are now identified we morphisms in this new category. This identification is desirable since morphisms behold the property of being a partial monoid i.e., two arrows can be composed iff the domain of one coincides with the codomain of the other. So we have solved the normalization issue, but we have encountered a new problem namely: the set Ξ of non zero

[4] $\mathbb{C}_*$ is just $\mathbb{C}$ without the zero element

vectors in $\mathcal{H}$ has no direct physical meaning. Isham noticed, though, that there exists a polar operation which takes you from subspaces of the Hilbert space, to strings of projector operators and vice versa. By using this polar operation and its properties (the details of which I leave up to the reader to find) it is possible to restrict the objects Ξ to a particular type of subset F of $\mathcal{H}$, such that the elements ϕ of F will have the property that they are "reducible" with respect to the strings belonging to the polar F^o of F i.e. $\forall\ \hat{Q} \in F^o$ and $\forall\ \phi \in F$, then $\hat{Q}\phi \neq 0$. This condition of reducibility becomes the context in which the elements of the category χ acquire physical meaning. Within this framework truth values of Quantum propositions of the form $A \in \Delta$, and subsequently the processo of state vector reduction is identified as follows:

$$[|\phi\rangle \in \mathcal{H}_{A\in\Delta}]_{\chi,\Xi} := \{Q \in Hom(\Xi\cdot)|\hat{Q}|\phi\rangle \in \hat{Q}\mathcal{H}_{A\in\Delta}\} \tag{1.8}$$

- The second method for solving the normalisation issue requires the introduction of a certain presheaf. I will not explain what a presheaf is and the properties it has, the interested reader should refer to Presheaves for further information.

  The presheaf that we are interested in (in this particular context) is the **reduction presheaf R** defined on the category $\widetilde{SP(\mathcal{H})}_0$. This category is essentially the same as the category $SP(\mathcal{H})_0$, previously defined, but with the difference that the arrows between two objects Q, and Q_1 have to be intended as a "reduction" of the string Q to the string Q_1, by an elimination of some projector operator, i.e. $Hom(Q, Q_1) := \{S \in SP(\mathcal{H})_0 | Q = Q_1 \star S\}$ where $\star$ is string concatenation. This definition of an arrow $Hom(Q, Q_1)$ implies that the string Q_1 can be derived from the string Q if we "right" divided Q by the string S, i.e. Q_1 is a "reduction" of the string Q by an amount S. The reason why Isham defines arrows in $\widetilde{SP(\mathcal{H})}_0$, as mentioned above, is because this definition implies the following: if $\hat{Q}|\psi\rangle = 0$ and $Q = Q_2 \star S$, then $\widehat{Q_2 \star S}|\psi\rangle = \hat{Q}_2\hat{S}|\psi\rangle = 0$, which implies that $\hat{S}|\psi\rangle = 0$. This property will enable us to overcome the normalization issues, as will be delineated later on.

Given this new category, roughly speaking, what the presheaf R does is to associate to each object Q in $\widetilde{SP(\mathcal{H})}_0$, the set of vectors which are reducible with respect to Q, and to each arrow $S \in Hom(Q, P)$ in $\widetilde{SP(\mathcal{H})}_0$ between two objects Q and P, the arrow between the respective subspaces of reducible vectors, i.e. $R(S) : R(Q) \to R(P)$ such that $R(S)|\psi\rangle := \hat{S}|\psi\rangle$.

In this context the State Vector reduction can be derived as follows: since

$$|\psi\rangle \overset{R(\hat{Q}_1)}{\rightarrow} \hat{Q}_1|\psi\rangle \overset{R(\hat{Q}_2)}{\rightarrow} \hat{Q}_2\hat{Q}_1|\psi\rangle \cdots \overset{R(\hat{Q}_q)}{\rightarrow} \hat{Q}_q\hat{Q}_{q-1}\cdots\hat{Q}_1|\psi\rangle \tag{1.9}$$

we can identify the State Vector reduction as

$$|\psi\rangle \overset{R(\hat{Q}_q)R(\hat{Q}_{q-1})\cdots R(\hat{Q}_1)}{\longrightarrow} \frac{\hat{Q}_q\hat{Q}_{q-1}\cdots\hat{Q}_1|\psi\rangle}{||\hat{Q}_q\hat{Q}_{q-1}\cdots\hat{Q}_1|\psi\rangle||} = \frac{\hat{S}|\psi\rangle}{||\hat{S}|\psi\rangle||} \tag{1.10}$$

The reason why, given this formulation, the normalisation issue does not occur can be easily understood with the aid of an example. Imagine we had two strings of projector operators:

$P_1 := \{\hat{Q}_1, \hat{Q}_2, \hat{Q}_3, \hat{Q}_4, \hat{Q}_5, \hat{Q}_6\}$ and $P_2 := \{\hat{Q}_1, \hat{Q}_2\}$.

We then define an arrow between them by $S := \{\hat{Q}_3, \hat{Q}_4, \hat{Q}_5, \hat{Q}_6\} :=$

$P_1 \rightarrow P_2$ such that $P_2 \star S = P_1$. The respective elements in the presheaf R are:

$R(P_1) := \{|\psi\rangle \in \mathcal{H}|\hat{P}_1|\psi\rangle \neq 0,$

$R(P_2) := \{|\phi\rangle \in \mathcal{H}|\hat{P}_1|\phi\rangle \neq 0,$

$R(S) : R(P_1) \rightarrow R(P_2)$, such that $R(S)|\psi\rangle = \hat{S}|\psi\rangle$.

If we then choose a vector $|\psi\rangle \in R(P_1)$, and applied the State Vector reduction procedure so to "reduce" it to a vector $\phi \in R(P_2)$, we would have to apply the string of projections S, so as to obtain an analogue of equation 1.10 where q=6 and $R(\hat{Q}_1)$ is replaced by $R(\hat{Q}_3)$. In order to do so we notice that the equality

$$\hat{P}_1|\psi\rangle = P_2 \star S|\psi\rangle \neq 0 \quad (1.11)$$

together with the above mentioned properties of morphisms in the category $\widetilde{SP(\mathcal{H})}_0$ imply that $S|\psi\rangle \neq 0$. Therefore, no normailsation problems occur.

Isham additionally defined the truth value of a Quantum Proposition within the Topos $\widetilde{SP(\mathcal{H})}_0$ as follows:

given a subset $\mathcal{H}_{A\in\Delta} \subset \mathcal{H}$, the truth value of the proposition $|\psi\rangle \in \mathcal{H}_{A\in\Delta}$ is

$$[|\psi\rangle \in \mathcal{H}_{A\in\Delta}]_{\widetilde{SP(\mathcal{H})}_0,Q} := \{S \in Hom(Q, \cdot)|\hat{S}|\psi\rangle \in \hat{S}\mathcal{H}_{A\in\Delta}\} \quad (1.12)$$

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