Topos Formulation of History Quantum Theory

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1.1 Why a History Theory?

The standard Copenhagen interpretation cannot describe closed systems, since the existence of an external observer is required.

This causes problems in any theory of quantum cosmology.

One significant recent attempt to deal with closed systems in quantum mechanics is history theory:

– *Omnes, Griffiths, Gell'mann & Hartle*: Histories as products of projection operators (therefore *not* projectors).

–*HPO formalism of consistent histories*: History propositions are identified with projection operators in bigger Hilbert space.

In this talk I will describe a new type of history quantum theory: a topos version of the temporal-logic part of the HPO formalism.

- A topos reformulation of quantum mechanics was put forward by Isham and Döring.

  An essential ingredient is a mapping of projection operators to certain objects in a topos.

- My aim is to extend this formulation to a *history* version of quantum theory, therefore my starting point is the HPO formalism.

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What is Topos Theory?

- A category is a collection of objects and a collection of 'maps' between these objects.

  The best-known example is *Sets*. But.......

- A topos is a category which is similar to *Sets*: fundamental mathematical properties (disjoint union, Cartesian product, etc) have a topos analogue. In particular

  – **Sub-object classifier** Ω: classifies subobjects in terms of what elements belong to them. Generalises the set $\{0, 1\}$ of truth-values in the category *Sets*.

  – Collection of all sub-objects of any object forms a **Heyting algebra**:

  A distributive algebra for which $S \vee \neg S \leq 1$. An internal logic, analogue to Boolean algebra in *Sets*

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- Kocken-Specher theorem $\Longrightarrow$ non-realist interpretation of quantum theory

  For quantum cosmology, need a reformulation of quantum theory which is 'more realist'. Isham, Butterfield & Döring: can be done through *topos theory*.

- Reformulate quantum theory to make it 'look like' classical physics:
  - Classical physics uses *Sets* as its mathematical structure.

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  - Logic of subsets in *Sets* is Boolean logic.

    Logic of subsets in a topos is a distributive logic

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- Need for *contexts* comes from K-S theorem: only within *abelian subalgebras* of $\mathcal{B}(\mathcal{H})$ can quantum theory 'look like' classical theory. *Contexts* form 'classical snapshots'.
  - The set of abelian subalgebras, $\mathcal{V}(\mathcal{H})$, forms a category under subset inclusion: $i_{V'V} : V' \subseteq V$

    i.e. consider all contexts at the same time!
  - **Example**: $V'' = V \cap V' \neq \emptyset$ then $\exists$ the inclusion maps $i_{V''V}$ and $i_{V''V'}$, therefore it is possible to 'relate' V and V'.
- Topos of presheaves over the category of abelian subalgebras : $Sets^{\mathcal{V}(\mathcal{H})^{op}}$.

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Let $\mathcal{C}, \mathcal{D}$ be categories. Then a presheaf is an assignment to each $\mathcal{D}$-object A of a $\mathcal{C}$-object $X(A)$, and to each $\mathcal{D}$-arrow $f : A \to B$ a $\mathcal{C}$-arrow $X(f) : X(B) \to X(A)$ such that:

- $X(1_A) = 1_{X(A)}$

- $X(f \circ g) = X(g) \circ X(f)$ for any $g : C \to A$

State spaces in physics

1. *Classical physics*: Physical quantity A represented $f_A : \mathcal{S} \to \mathbb{R}$.
2. *Quantum physics*: Physical quantity A represented $\hat{A} : \mathcal{H} \to \mathcal{H}$.
3. *Topos physics*: *Spectral presheaf* $\underline{\Sigma} : \mathcal{V}(\mathcal{H}) \to$ *Set* such that

   $V \mapsto \underline{\Sigma}_V := \{$simultaneous eigenvectors of $V\}$; i.e., the possible values of the physical quantities in V.

   Given $\hat{A}$, then $f_{\hat{A}} : \underline{\Sigma} \to \mathbb{R}$!

$\underline{\Sigma}$ is the '**state object**' of the theory.

Propositions

1. *Classical physics*:

$$\text{“}A \in \Delta\text{”} \rightarrow f_A^{-1}(\Delta) = \{s \in \mathcal{S} | f_A(s) \in \Delta\} \subseteq \mathcal{S}$$

2. *Quantum theory*:

$$\hat{P} = \hat{E}[A \in \Delta] \in P(\mathcal{H})$$

3. *Topos physics*: Need to identify $\hat{P}$ with sub-object of $\underline{\Sigma}$; i.e., for each V need subset of $\underline{\Sigma}_V$; i.e., a *projection operator* in V.

- **'Daseinisation':**

$$\delta : P(\mathcal{H}) \to P(V)$$
$$P \mapsto \delta(\hat{P})_V$$

where $\delta(\hat{P})_V := \bigwedge\{\hat{\alpha} \in P(V) | \hat{\alpha} \geq \hat{P}\}$: the 'best' approximation of $\hat{P}$ (from above) by projectors in V.

- **Relation to** $Sub(\underline{\Sigma})$:

Any projector in V gives a subset of $\underline{\Sigma}_V$. Therefore get map, for each V, $\hat{P} \mapsto \underline{\delta(\hat{P})}_V$.

Can show that this corresponds to $\delta : P(\mathcal{H}) \to Sub(\underline{\Sigma})$!

Thus δ maps propositions about a quantum system to a *distributive* lattice in a *contextual* manner.

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- **States in classical physics**

  In classical physics a microstate is a point in the state space.

- **States in the topos formulation of quantum theory**
  - $\underline{\Sigma}$ has no 'points' $\Longrightarrow$ Equivalent to the K-S theorem
  - Topos analogues of a state is a (non-point!) sub-object of the state object $\underline{\Sigma}$:

  Pseudo-state: $\underline{\mathfrak{w}}^{|\psi\rangle} := \underline{\delta(|\psi\rangle\langle\psi|)} \subseteq \underline{\Sigma}$,

$$\delta(\,|\psi\rangle\langle\psi|)_V := \bigwedge\{\hat{\alpha} \in P(V) \mid |\psi\rangle\langle\psi| \leq \hat{\alpha}\}$$

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- **Truth values in classical physics**

  A proposition $Q \subseteq \mathcal{S}$ is true in a state s iff $s \in Q$.
  This is equivalent to $\{s\} \subseteq Q$.

- **Truth values in the topos formulation of quantum theory**
  - In topos quantum theory, we define the proposition $\underline{\delta(\hat{P})}$ to be 'totally true' given the pseudo state $\underline{\mathfrak{w}}^{|\psi\rangle}$ iff $\underline{\mathfrak{w}}^{|\psi\rangle} \subseteq \underline{\delta(\hat{P})}$. This means that, for all V, $\mathfrak{w}_V^{|\psi\rangle} \leq \delta(\hat{P})_V$.
  - *However*, in a topos a proposition can be 'partially true' using 'contextual truth values'. At stage V we define

$$v(\underline{\mathfrak{w}}^{|\psi\rangle} \subseteq \underline{\delta(\hat{P})})_V := \{V' \subseteq V \mid \underline{\mathfrak{w}}_{V'}^{|\psi\rangle} \subseteq \underline{\delta(\hat{P})}_{V'}\}$$
$$= \{V' \subseteq V \mid \langle\psi|\delta(\hat{P})_{V'}|\psi\rangle = 1\}$$

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– This is called a *sieve* on V.

  The collection, $\underline{\Omega}_V$, of all sieves on V forms a **Heyting algebra**!

– For varying V such truth values form a global section $\Gamma\underline{\Omega}$ of the sub-object classifier $\underline{\Omega}$.

The set $\Gamma\underline{\Omega}$ is itself a Heyting algebra!

– Thus we have a Heyting algebra of propositions *and* a Heyting algebra of 'generalised' truth values!

4.1 HPO Formulation of Quantum Temporal Logic

- Identify the set of all history propositions with projection operators in a new Hilbert space $\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2} \otimes \cdots \otimes \mathcal{H}_{t_n}$
  - 'Homogeneous histories': Tensor products of projection operators
    $$\alpha := \hat{\alpha}_{t_1} \otimes \hat{\alpha}_{t_2} \otimes \cdots \otimes \hat{\alpha}_{t_n}$$
  - 'Inhomogeneous histories':
    $$\begin{aligned} \neg(\hat{\alpha}_{t_1} \otimes \hat{\alpha}_{t_2}) &:= \hat{1}_{\mathcal{H}_{t_1}} \otimes \hat{1}_{\mathcal{H}_{t_2}} - \hat{\alpha}_{t_1} \otimes \hat{\alpha}_{t_2} \\ &= (\neg\hat{\alpha}_{t_1} \otimes \hat{\alpha}_{t_2}) \vee (\hat{\alpha}_{t_1} \otimes \neg\hat{\alpha}_{t_2}) \vee (\neg\hat{\alpha}_{t_1} \otimes \neg\hat{\alpha}_{t_2}) \end{aligned}$$
  - 'Type III' propositions

Entangled state:

$$|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle$$

Associated projection operator

$$\hat{P}_{\text{entangled}} = (|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle)(\langle\uparrow|\langle\downarrow| - \langle\downarrow|\langle\uparrow|)$$

$$\hat{P}_{\text{entangled}} \neq \hat{P}_{ud} \vee \hat{P}_{du} = (|\uparrow\rangle|\downarrow\rangle)(\langle\downarrow|\langle\uparrow|) \vee (|\downarrow\rangle|\uparrow\rangle)(\langle\uparrow|\langle\downarrow|)$$

Logical entanglement given by inhomogeneous propositions is not sufficient to account for the full quantum entanglement.

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  - 'Type III' propositions
- I.e., the tensor product implements the 'and then' temporal connective in *quantum temporal logic*.

- **Aim**: Find a *topos* representation of the homogeneous history $\alpha = (A_1 \in \Delta_1)_{t_1} \sqcap (A_2 \in \Delta_2)_{t_2} \cdots \sqcap (A_n \in \Delta_n)_{t_n}$
  - Individual-time propositions are identified with sub-objects of the spectral presheaf $\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_i})}$. Collection of all such sub-objects, $Sub(\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_i})})$, forms a Heyting algebra.
  - Temporal structure of Heyting algebras requires tensor product of Heyting algebras? Yes!
- I will identify homogeneous histories with elements in $Sub(\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_1})}) \otimes Sub(\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_2})}) \otimes \cdots \otimes Sub(\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_n})})$

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  The definition of the tensor product of Heyting algebras allows for both *homogeneous* and *inhomogeneous* propositions.

Given two Heyting algebras A and B the tensor product $A \otimes B$ is defined as the Heyting algebra generated by elements $a \otimes b$ for all $a \in A$ and $b \in B$ subject to

- $(a_1 \otimes b_1) \wedge (a_2 \otimes b_2) := (a_1 \bigwedge a_2) \otimes (b_1 \bigwedge b_2)$
- $(a_1 \otimes b) \vee (a_2 \otimes b) := (a_1 \bigvee a_2) \otimes b$
- $(a \otimes b_1) \vee (a \otimes b_2) := a \otimes (b_1 \vee b_2)$

Therefore

$$(a_1 \vee b_1) \otimes (a_2 \vee b_2) = (a_1 \otimes a_2) \vee (a_1 \otimes b_2) \vee (b_1 \otimes a_2) \vee (b_1 \otimes b_2) \geq a_1 \otimes a_2 \vee b_1 \otimes b_2$$

- We need to relate the Heyting algebra $Sub(\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_1})}) \otimes Sub(\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_2})})$ to sub-objects of some 'state object' in some topos related to quantum theory.

$$\hat{\alpha}_{t_1} \to \underline{\delta(\hat{\alpha}_{t_1})} \in Sub(\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_1})}) \in Sets^{\mathcal{V}(\mathcal{H}_{t_1})^{\mathrm{op}}}$$

$$\hat{\alpha}_{t_2} \to \underline{\delta(\hat{\alpha}_{t_2})} \in Sub(\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_2})}) \in Sets^{\mathcal{V}(\mathcal{H}_{t_2})^{\mathrm{op}}}$$

- To what topos does the history proposition $\underline{\delta(\hat{\alpha}_{t_1})} \otimes \underline{\delta(\hat{\alpha}_{t_2})} \in Sub(\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_1})}) \otimes Sub(\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_2})})$ belong?

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Intermediate topos: $\mathit{Sets}^{\mathcal{V}(\mathcal{H}_{t_1})^{\mathrm{op}}\times\mathcal{V}(\mathcal{H}_{t_2})^{\mathrm{op}}}$

$$\begin{aligned} p_1 : \mathcal{V}(\mathcal{H}_{t_1}) \times \mathcal{V}(\mathcal{H}_{t_2}) &\to \mathcal{V}(\mathcal{H}_{t_1}) \\ \langle V_1, V_2\rangle &\mapsto V_1 \\ p_2 : \mathcal{V}(\mathcal{H}_{t_1}) \times \mathcal{V}(\mathcal{H}_{t_2}) &\to \mathcal{V}(\mathcal{H}_{t_2}) \\ \langle V_1, V_2\rangle &\mapsto V_2 \end{aligned}$$

from which we can obtain

$$\begin{aligned} p_1^* &: \mathit{Sets}^{\mathcal{V}(\mathcal{H}_{t_1})^{\mathrm{op}}} \to \mathit{Sets}^{\mathcal{V}(\mathcal{H}_{t_1})^{\mathrm{op}}\times\mathcal{V}(\mathcal{H}_{t_2})^{\mathrm{op}}} \\ p_2^* &: \mathit{Sets}^{\mathcal{V}(\mathcal{H}_{t_2})^{\mathrm{op}}} \to \mathit{Sets}^{\mathcal{V}(\mathcal{H}_{t_1})^{\mathrm{op}}\times\mathcal{V}(\mathcal{H}_{t_2})^{\mathrm{op}}} \end{aligned}$$

which gives the well-defined product in $\mathit{Sets}^{\mathcal{V}(\mathcal{H}_{t_1})^{\mathrm{op}}\times\mathcal{V}(\mathcal{H}_{t_2})^{\mathrm{op}}}$

$$\begin{aligned} (\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_1})} \times \underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_2})})_{\langle V_1, V_2\rangle} &:= (p_1^*(\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_1})}) \times p_2^*(\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_2})}))_{\langle V_1, V_2\rangle} \\ &= \underline{\Sigma}_{V_1}^{\mathcal{V}(\mathcal{H}_{t_1})} \times \underline{\Sigma}_{V_2}^{\mathcal{V}(\mathcal{H}_{t_2})} \end{aligned}$$

Theorem: $Sub(\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_1})}) \otimes Sub(\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_2})}) \cong Sub\big(\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_1})} \times \underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_2})}\big)$

- **Conjecture:** Proposition $\alpha_1 \sqcap \alpha_2$ should be represented by

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- But, the HPO-representative, $\hat{\alpha}_1 \otimes \hat{\alpha}_2$, of the history proposition $\alpha_1 \sqcap \alpha_2$ belongs to $\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2}$ and is daseinised by

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  $\mathcal{V}(\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2})$ contains *entangled contexts* $W = V_1 \otimes V_2 + V_3 \otimes V_4$ and so $\mathbf{Sets}^{\mathcal{V}(\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2})^{\text{op}}} \neq \mathbf{Sets}^{\mathcal{V}(\mathcal{H}_{t_1})^{\text{op}} \times \mathcal{V}(\mathcal{H}_{t_2})^{\text{op}}}$

- Want to find a relation between $\underline{\delta(\hat{\alpha}_1)} \otimes \underline{\delta(\hat{\alpha}_2)} \in p_1^*(\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_1})}) \times p_2^*(\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_2})})$ and $\underline{\delta(\hat{\alpha}_1 \otimes \hat{\alpha}_2)} \in \underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2})}$.

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  $\mathcal{V}(\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2})$ contains *entangled contexts* $W = V_1 \otimes V_2 + V_3 \otimes V_4$ and so $\mathbf{Sets}^{\mathcal{V}(\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2})^{\mathrm{op}}} \neq \mathbf{Sets}^{\mathcal{V}(\mathcal{H}_{t_1})^{\mathrm{op}} \times \mathcal{V}(\mathcal{H}_{t_2})^{\mathrm{op}}}$

- Want to find a relation between $\underline{\delta(\hat{\alpha}_1)} \otimes \underline{\delta(\hat{\alpha}_2)} \in p_1^*(\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_1})}) \times p_2^*(\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_2})})$ and $\underline{\delta(\hat{\alpha}_1 \otimes \hat{\alpha}_2)} \in \underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2})}$.

Must find a relation between **Sets**$^{\mathcal{V}(\mathcal{H}_{t_1})^{\text{op}} \times \mathcal{V}(\mathcal{H}_{t_2})^{\text{op}}}$ and **Sets**$^{\mathcal{V}(\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2})^{\text{op}}}$

- Relation between context categories:

$$\begin{aligned} \theta : \mathcal{V}(\mathcal{H}_{t_1}) \times \mathcal{V}(\mathcal{H}_{t_2}) &\rightarrow \mathcal{V}(\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2}) \\ \langle V_1, V_2 \rangle &\mapsto V_1 \otimes V_2 \end{aligned}$$

- Induced relation between topoi:

$$\theta^* : \textbf{Sets}^{\mathcal{V}(\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2})^{\text{op}}} \rightarrow \textbf{Sets}^{\mathcal{V}(\mathcal{H}_{t_1})^{\text{op}} \times \mathcal{V}(\mathcal{H}_{t_2})^{\text{op}}}$$

$$\begin{aligned} (\theta^* \underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2})})_{\langle V_1, V_2 \rangle} &:= (\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2})})_{\theta(\langle V_1, V_2 \rangle)} = \underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2})}_{V_1 \otimes V_2} \\ &\cong \underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_1})}_{V_1} \times \underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_2})}_{V_2} \end{aligned}$$

using the fact that, for contexts of the form $V_1 \otimes V_2$,

$$\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2})}_{V_1 \otimes V_2} \cong \underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_1})}_{V_1} \times \underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_2})}_{V_2}$$

Conclusion: To account for both homogeneous and inhomogeneous ('logically entangled') propositions the *intermediate* topos $\mathbf{Sets}^{\mathcal{V}(\mathcal{H}_{t_1})^{\mathrm{op}}\times\mathcal{V}(\mathcal{H}_{t_2})^{\mathrm{op}}}$ suffices.

But, in full topos $\mathbf{Sets}^{\mathcal{V}(\mathcal{H}_{t_1}\otimes\mathcal{H}_{t_2})^{\mathrm{op}}}$ there also arise (i) entangled *contexts* $W = V_1 \otimes V_2 + V_3 \otimes V_4$; and (ii) a third type that cannot be expressed in this way. The physical significance of these needs to be explored further.

Claim: because of the absence of state-vector reduction in the topos approach, it is meaningful to define the truth value of a (homogeneous) history proposition in terms of the truth values of the individual time components:

$$v\big((A_1 \in \Delta_1)_{t_1} \sqcap (A_2 \in \Delta_2)_{t_2};\ |\psi\rangle_{t_1}\big) := v\big(A_1 \in \Delta_1;\ |\psi\rangle_{t_1}\big) \otimes v\big(A_2 \in \Delta_2;\ |\psi\rangle_{t_2}\big)$$

where $|\psi\rangle_{t_2} = \hat{U}(t_2, t_1)\,|\psi\rangle_{t_1}$.

Want to find a ‘topos interpretation’ of the above equation.

Problem: truth values belong to different topoi

$$v\left(A_1 \in \Delta_1;\, |\psi\rangle_{t_1}\right) := v(\underline{\mathfrak{w}}^{|\psi\rangle_{t_1}} \subseteq \underline{\delta(\hat{P}_1)}) \in \Gamma\underline{\Omega}^{\mathcal{V}(\mathcal{H}_{t_1})}$$

$$v\left(A_2 \in \Delta_2;\, |\psi\rangle_{t_2}\right) := v(\underline{\mathfrak{w}}^{|\psi\rangle_{t_2}} \subseteq \underline{\delta(\hat{P}_2)}) \in \Gamma\underline{\Omega}^{\mathcal{V}(\mathcal{H}_{t_2})}$$

Solution: pull back to the 'intermediate topos' $\mathbf{Sets}^{\mathcal{V}(\mathcal{H}_{t_1})^{\mathrm{op}} \times \mathcal{V}(\mathcal{H}_{t_2})^{\mathrm{op}}}$.
In $\mathbf{Sets}^{\mathcal{V}(\mathcal{H}_{t_1})^{\mathrm{op}} \times \mathcal{V}(\mathcal{H}_{t_2})^{\mathrm{op}}}$ we have:

$$v\left(\underline{\mathfrak{w}}^{|\psi\rangle_{t_1}} \subseteq \underline{\delta(\hat{P}_{t_1})}\right) \otimes v\left(\underline{\mathfrak{w}}^{|\psi\rangle_{t_2}} \subseteq \underline{\delta(\hat{P}_{t_2})}\right) \cong v\left(\underline{\mathfrak{w}}^{|\psi\rangle_{t_1}} \otimes \underline{\mathfrak{w}}^{|\psi\rangle_{t_2}} \subseteq \underline{\delta(\hat{P}_{t_1})} \otimes \underline{\delta(\hat{P}_{t_2})}\right)$$

using the fact that (theorem)

$$\boxed{\Gamma\underline{\Omega}^{\mathcal{V}(\mathcal{H}_{t_1})} \otimes \Gamma\underline{\Omega}^{\mathcal{V}(\mathcal{H}_{t_2})} \simeq \Gamma\underline{\Omega}^{\mathcal{V}(\mathcal{H}_{t_1}) \times \mathcal{V}(\mathcal{H}_{t_2})}}$$

Therefore, so long as entangled contexts (plus type III) are not considered it is possible to define truth values of history propositions as tensor products of truth values of individual-time propositions.

In particular, to obtain a topos formulation of quantum history theories the *intermediate* topos suffices.

- The aim was to obtain a topos formulation of the temporal quantum logic in the HPO formalism.

  The strategy adopted was to extend the topos formulation of quantum mechanics put forward by Isham and Döring so as to include temporal propositions.

- I have been able to represent both *homogeneous* and *inhomogeneous* history propositions as elements of the Heyting algebra $Sub(\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_1})}) \otimes Sub(\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_2})})$

  $Sub(\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_1})}) \otimes Sub(\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_2})}) \in \mathbf{Sets}^{\mathcal{V}(\mathcal{H}_{t_1})^{\mathrm{op}} \times \mathcal{V}(\mathcal{H}_{t_2})^{\mathrm{op}}}$; i.e., no entangled *contexts*

- The topos that represents the full HPO formalism is the topos $\mathbf{Sets}^{(\mathcal{V}(\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2}))^{\mathrm{op}}}$ which includes entangled and type III contexts.

- For non-entangled contexts the two topoi coincide.

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Future Work

1. Is it possible to represent, with a novel mathematical structure, *type III propositions* in topos-theoretical terms?
   - Logical entanglement is captured by the notion of tensor product of Heyting algebra.
   - Quantum entanglement *might* be captured by the notion (yet to be defined) of 'quantum' tensor product.
2. Extend the topos reformulation of history theory to the case of *continuous* time.

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2. Extend the topos reformulation of history theory to the case of *continuous* time.

Every frame A defines a complete Heyting algebra (cHa) in such a way that the operations $\wedge$ and $\vee$ are preserved, and the implication relation $\rightarrow$ is defined as follows

$$a \rightarrow b = \bigvee \{c : c \wedge a \leq b\}$$

Frame distributivity implies that $(a \rightarrow b) \wedge a \leq b$, from which it follows

$$c \leq a \rightarrow b \qquad \text{iff} \qquad c \wedge a \leq b$$