Kochen-Specker Theorem and Valuations in the language of Topos Theory

Cecilia Flori

Imperial Collage of Science, Technology & Medicine

South Kensington

London SW7 2BZ

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Abstract

Quantum Theory seems to be an anti-realist Theory, because of the existance of the Kochen- Specker Theorem which asserts that it is impossible to evaluate propositions such that their truth values belong to the set $\{1, 2\}$. However, it is possible, using the Topos Theory of presheaves, to define a new kind of valuation (sieved-valued valuations) for Quantum quantities which is defined on all operators. The truth values that this valuation assignes, though, are contextual and multivalued, in agreement with the mathematical formalism of the theory. What we propose in this paper is a delineation and discussion of the above idea put foward by C.J.Isham and J.Butterfield. In particular, we begin with a comparison between Classical Boolean Logic and the non-Boolean Logic of Quantum Mechanics, we then intoduce the Topos Theory of presheafs and its implementation in defining a valuation for Quantum propositions. Our mane focus is to try and motivate the choice of sieved-valued valuations using the mathematical formalism of Quantum Theory itself.

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Chapter 1

Introduction

"Useful as it is under everyday circumstances to say that the world exists "out there" independent of us, that view can no longer be upheld. There is a strange sense in which this is a "participating universe"", Wheeler (1983).

The above statement reveals the radical difference that exists between the view of the world given by Quantum Mechanics and the view given by Classical Physics. In fact, the existence of an "objective external world", which is postulated by Classical Physics, seems to be rejected by Quantum Mechanics. The cause of this interpretative differences between the two theories can be traced back to the different algebras used to relate propositions[1]. Precisely the propositions in Classical Physics form a Boolean algebra, while propositions in Quantum Mechanics form a non-Boolean algebra. This feature of Quantum Mechanics entails that properties can not be said to be possessed by a system, denying, in such a way, the existence of an independent "outside world". An attempt to

[1] Propositions are defined as statements regarding properties of a given system.

give Quantum Mechanics the status of a realist[2] theory is given by the hidden variable theories, which postulate the following:

- in any state $\vec{\psi}$ observables A posses an "objectively existing value"
- values of observables A are determined by $\vec{\psi}$ and hidden variables.

However this theories where disproved by the Kochen-Specker Theorem and Bell inequalities, both of which show that properties are not possessed by a Quantum System, therefore rejecting the first assumption of the Hidden variable theories. In particular, the Kochen-Specker theorem asserts that it is impossible to evaluate propositions regarding values possessed by physical entities represented by projection operators, such that their truth values belong to the set $\{1, 2\}$, therefore depriving of meaning any statement regarding a state of affairs of a system, since generally speaking, a statement is said to be meaningful if its validity can be assessed. The Bell inequalities go further and show the impossibility of a local[3] realist interpretation of Quantum Mechanics.

Do we then have to accept that Quantum theory is a non-realist theory and therefore regard any statement about states of affair of a system as meaningless, or is there a way of reformulating the Kochen-Specker theorem and the Bell inequalities so as to give Quantum theory a realist flavor?

In this paper we will concentrate on the Kochen-Specker Theorem, in particular we will introduce and justify, the redefinition of the Kochen-Specker Theorem

[2]A realist theory is a theory in which the following conditions are satisfied 1) propositions are related through a Boolean algebra 2) propositions can always be assessed to be either true or false.

[3]Locality means that, given a composite system, the value of a physical quantity of an individual constituent of the system, is independent on what is measured on any other constituent.

in terms of Topos Theory, proposed by C.J.Isham and J. Butterfield [1], [9], [11] [8]. According to the authors, by introducing a new kind of logic for Quantum propositions defined in terms of Topos Theory, it is then possible to re-defining the Kochen-Specker Theorem in such a way that it becomes meaningful to ask whether a proposition in Quantum Mechanics is true or false, i.e. it is possible to assign truth values to propositions. Never the less this approach does not render Quantum Theory a completely realist theory since the truth values of Quantum propositions it proposes, are multi-valued and contextual, giving rise to a non-Boolean Logic of propositions. This entails that a valuation in Quantum Theory can not be a simple bijective function from the state space to the Reals as it is in Classical Physics. Therefore, propositions can not be thought of as being possessed by the system. So, even though the Topos approach to the Kochen-Specker Theorem enables us to give meaning to statements about properties "of" a system, this meaning is contextual, it depends on what other properties are being considered at the same time. In this way Quantum Theory becomes a "neo-realist" [4] theory, i.e. a theory half way in between being a realist theory and an anti-realist theory. The advantage of this approach is that the logic of Quantum propositions it proposes is distributive, therefore it can be used as a deductive system of reasoning. Moreover, different from any other multi- valued type logic, it enables to define the logical connectives in an unambiguous way, and such that the metalanguage[5]/object-language [6] distinction is not violated.

[4] name created by C.J.Isham and J Butterfield [1], [9], [11], [8]

[5] Metalanguage is the language used to make statements about another language.

[6] Object-language is the language which is being studied through the Metalanguage.

This paper develops in the following way: in Chapter 2 we analyse the concept of property of a system which arises from the Mathematical Formalism of Classical Theory. In particular we show that, in Classical Physics, a valuation of a proposition is simply a function from the State Space of the system to the Reals, giving rise to a Boolean algebra of Propositions. This approach entails a realistic view of the world since properties are considered as being possessed by the system.

In Chapter 3 we show how the Mathematical Formalism of Quantum Mechanics entails a non Boolean algebra of propositions, and how truth values of propositions are given in terms of propositions belonging or not to a certain subset of the Hilbert space.

In Chapter 4 we state the Kochen-Specker Theorem and show how one of the assumptions of the theorem, namely the functional composition principle, arises from three assumptions, which are then rejected by the Quantum Mathematical Formalism itself.

In chapter 5 we motivate the choice of Topos Theory as the tool to define a Logical system in Quantum Mechanics and we describe how the Kochen-Specker Theorem can be expressed in the language of Topos.

In Chapter 6 we describe two different kinds of valuations in Quantum Theory that arise from Topos Theory.

The remaining section is a delineation of a concrete examples of these new valuations that arise in Quantum Mechanics.

Most of the material used in this paper has been taken from the works of Professors C.J.Isham and J. Butterfield [1], [9], [11],[2], [8] which I have interpreted and re-elaborated according to my own understanding of the subject.

Chapter 2

Classical Physics

The mathematical formalism of classical mechanics leads to a conception of the world which is realistic and deterministic. In fact, according to classical physics, it is meaningful to talk about objects possessing properties, and it is possible to determine, with certainty, how these properties will evolve in time. This view of the world is in accordance with common believes about reality produced by daily observation. In what follows, we delineate the basic ideas behind Classical Mechanics and how these ideas give rise to a realist and deterministic view of the world. The postulates of classical mechanics are the following.

1. The possible configurations (how things are) of a system are represented by elements of the state space S i.e. there is a one to one correspondence between possible states of the system and elements of the state space. Knowing the state of a system at a particular point in time allows one to know the state of the system at an earlier time or a later time i.e. classical mechanics is entirely deterministic.

2. A physical quantity is defined as a real valued function

$$f_A : S \to \mathbb{R} \tag{2.1}$$

from the state space S to the real numbers such that $f_A(s)$ represents the value of the physical quantity A, if the system is in state s, that is a member of S.

3. A value $V : S \to \mathbb{R}$ is a function from the state space to the reals such that $V_A(s)$ represents the value of the physical quantity A if the system is in the sate s that is a member of S. Clearly, from postulate 2 it follows that $V_A(s) = f_A(s)$.

   An important condition that the value function should follow is the **functional composition condition** (FUNC): given a physical quantity A whose value is V(A) and a function $f : \mathbb{R} \to \mathbb{R}$ such that f(A) is another physical quantity, then the following equality holds:

$$V(f(A)) = f(V(A))$$

   i.e. the value of the function f of the physical quantity A is equivalent to the function f evaluated on the value of the physical quantity A. This implies that measurement can be predicted with certainty.

4. Propositions (statement about properties possessed by a system) are identified with subsets of the state space S.

   Consider a proposition of the form "$A \in \Delta$" where (A) represents a physical quantity and (Δ) represents the set of its possible values. Since a quantity A is defined as a real valued function $f_A : S \to \mathbb{R}$ then ($A \in \Delta$) is identified with those subsets of the state space S which correspond to configurations of the system in which the above proposition is true, i.e. proposition $(A \in \Delta)$ corresponds to the subset $f_A^{-1}\{\Delta\}$ of S. This means that subsets of S represents equivalent classes of propositions i.e. $S_{A\in\Delta} = \{s \in S | A(s) \in \Delta\}$ represents those subsets of S in which the proposition $(A \in \Delta)$ is true.

For each subset $S_{A\in\Delta} : \{s \in S | A(s) \in \Delta\}$ of S, corresponds an associated characteristic function $\chi_{A\in\Delta} : S \to \mathbb{R}$ such that :

$$\chi_{A\in\Delta}(s) = \begin{cases} 1 & if \quad f_A(s) \in \Delta \\ 0 & otherwise \end{cases}$$

where 1=true and 0=false.

So in classical physics it emerges that the truth values of propositions are the elements of the Boolean set {1,0}.

5. The set of these subsets forms a Boolean algebra (Definition 2.1) and, therefore, defines a logic for propositions regarding properties of a system in classical physics.

A consequence of postulate 2 is that, to each subset s of S corresponds a unique value for any physical quantity described in that system, i.e. the function $f_A : S \to \mathbb{R}$ for any physical quantity A is one to one. This entails that properties are actually possessed by a system.

2.1 Boolean Algebra

In this section we will analyse how the Boolean algebra of subsets of the state space S can be used to define a logical System for classical propositions.

Definition 2.1 *A **boolean algebra** S is an orthocomplemented and distributive lattice.*

The property of being *orthocomplemented* implies that there exists a maximum S wich corresponds to the whole algebra, and a minimum $\emptyset$ and, for any two elements S_1 and S_2 that belong to S, the following relations are satisfied:

1. $(S_1^c)^c = S_1$

2. $S_1 \subseteq S_2$ implies $S_2^c \subseteq S_1^c$

3. $S_1 \cup S_1^c = 1$ and $S_1 \cap S_1^c = 0$

where $S_i^c = S - S_i$ defines the complement of S_i. The property of being *distributive* implies :

i) $S_1 \cap (S_2 \cup S_3) = (S_1 \cap S_2) \cup (S_1 \cap S_3)$

ii) $S_1 \cup (S_2 \cap S_3) = (S_1 \cup S_2) \cap (S_1 \cup S_3)$

Postulate 4 and 5 in the previous section, imply that the logic of classical propositions can be represented by the Boolean algebra of the subsets of S where, partial ordering is given in terms of subset inclusion and the logical connectives are represented by set theoretic operation.

Denoting "and", "or","implies", "not", "iff" with $\wedge$,$\vee$,$\Longrightarrow$, $\neg$,$\Longleftrightarrow$ we get:

$$(A \in \Delta) \wedge (B \in \Theta) \longleftrightarrow S_{A\in\Delta} \cap S_{B\in\Theta}$$

$$(A \in \Delta) \vee (B \in \Theta) \longleftrightarrow S_{A\in\Delta} \cup S_{B\in\Theta}$$

$$(A \in \Delta) \Longrightarrow (B \in \Theta) \longleftrightarrow S_{A\in\Delta} \subset S_{B\in\Theta}$$

$$\neg(A \in \Delta) \longleftrightarrow S - S_{A\in\Delta} = \{s \in S | s \notin S_{A\in\Delta}\} = S_{\neg(A\in\Delta)} = S^c_{A\in\Delta}$$

$$(A \in \Delta) \Longleftrightarrow (B \in \Theta) \longleftrightarrow S_{A\in\Delta} = S_{B\in\Theta}$$

$$1 \longleftrightarrow S$$

$$0 \longleftrightarrow \emptyset$$

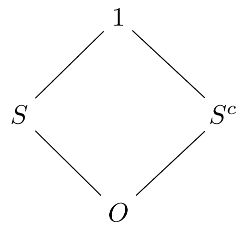
Where the unit element 1 corresponds to propositions that are always true and the zero element 0 corresponds to the null proposition that is always false.

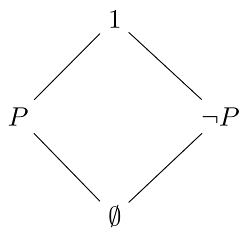
Writing proposition $(A \in \Delta)$ as P_A and proposition $(B \in \Theta)$ as P_B we have the following [4].

Definition 2.2 *$P_A \vee P_B$, $(S_{P_A} \cup S_{P_B})$ corresponds to the least upper bound (suprimum, join) of P_A and P_B, (S_{P_A}, S_{P_B})*
i.e. the unique element above P_A and P_B (S_{P_A}, S_{P_B}) with respect to the partial ordering such that there is no other element in between

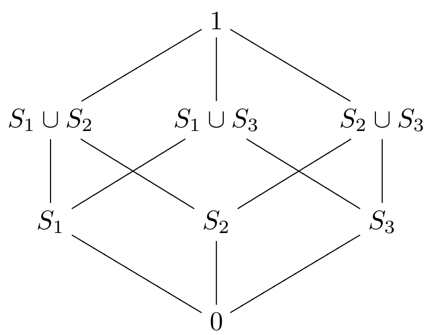
Definition 2.3 *$P_A \wedge P_B$, $(S_{P_A} \cap S_{P_B})$ corresponds to the greatest lower bound (infimum, meet) of P_A and P_B, (S_{P_A}, S_{P_B})*
i.e. the unique element below P_A and P_B, (S_{P_A}, S_{P_B}) with respect to the partial ordering such that there is no other element in between.

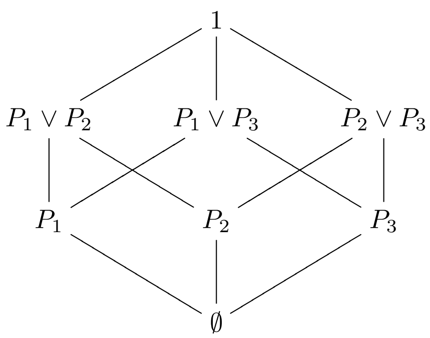
Some examples of Boolean lattices are: diagram $\mathbb{B}_1$ [10]





and diagram $\mathbb{B}_2$ [10]





(where the lines represent the partial ordering)

From the above diagrams we can see that the state space S gets divided into subsets which encompass all the possible configurations of the system such that only one of them can occur at a given time. Let us consider, now, diagram $\mathbb{B}_1$, according to which the possible states of a system correspond to the subsets: $\{S_1, S_1 \cup S_2, S_1 \cup S_3, 1\}$ $\{S_2, S_1 \cup S_2, S_2 \cup S_3, 1\}$ and $\{S_3, S_1 \cup S_3, S_2 \cup S_3, 1\}$, where each subset corresponds to propositions about the system which can be true at the same time. Since Boolean lattices are formed by sublattices, one can change from one lattice to another according to how the state space is divided. An example may help to clarify. If we consider diagrams $\mathbb{B}_1$ and $\mathbb{B}_2$, in $\mathbb{B}_1$ the state space S has been divided into S_1 and S_1^c whereas in $\mathbb{B}_2$ the state space has been divided into S_1, S_2 and S_3 where S_2 and S_3 represent the complement of S_1. In order to go from $\mathbb{B}_1$ to $\mathbb{B}_2$ one needs to create a homomorphism $h : \mathbb{B}_1 \rightarrow \mathbb{B}_2$ such that the inner structure is respected i.e. the following equations are satisfied:

$$h(P_1 \vee P_2) = h(P_1) \vee h(P_2)$$

$$h(P_1 \wedge P_2) = h(P_1) \wedge h(P_2)$$

$$h(\neg P_1) = \neg(h(P_1))$$

From these equations it follows that minimum and maximum elements of the domain lattice get mapped into the minimum and maximum of the codomain lattice. In the example above a possible candidate for

$$h : \mathbb{B}_1 \rightarrow \mathbb{B}_2$$

could be:

$$h(P) = P_1$$

$$h(\neg P) = P_2 \vee P_3 = \neg(P_1)$$

while a homeomorphism

$$g : \mathbb{B}_2 \rightarrow \mathbb{B}_1$$

could be :

$$g(P_1) = g(P_2) = g(P_1 \vee P_2) = P$$

$$g(P_3) = g(P_1 \vee P_3) = g(P_2 \vee P_3) = \neg P$$

One way of proving that the above homomorphism do indeed respect the inner structure is to prove the following:

1. $h(1_{\mathbb{B}_1}) = 1_{\mathbb{B}_2}$

   Proof 2.1

   $$h(1_{\mathbb{B}_1}) = h(P \vee \neg(P)) = h(P) \vee h(\neg(P)) = P_1 \vee \neg(P_1) = 1_{\mathbb{B}_2}$$

2. $g(1_{\mathbb{B}_2}) = 1_{\mathbb{B}_1}$

   Proof 2.2

   $$g(1_{\mathbb{B}_2} = g(P_1 \vee \neg(P_1)) = g(P_1) \vee g(\neg(P_1)) = g(P_1) \vee \neg(g(P)) = P \vee \neg P = 1_{\mathbb{B}_1}$$

An important type of homomorphism is the 2-valued homeomorphism on a boolean algebra $\mathbb{B}$ i.e. the map $f : \mathbb{B} \rightarrow \{0,1\}$, (where $\{0,1\}$ is a boolean algebra). This 2-valued homeomorphism assigns the value 1 to one and only one atom (the smallest element greater than the null element) of the lattice and 0 to the rest. Therefore in classical physics $f : \mathbb{B} \rightarrow \{0,1\}$ represents a truth value assignment to propositions where 1=true and 0=false.

Chapter 3

Quantum Mechanics

As we have seen, the Mathematical Formalism of Classical Physics entails a precise and well defined view of reality, which, in turn, entails a Boolean logic. Unfortunately this is not the case for quantum mechanics, in fact, as we shall see later on, the logic that the quantum mechanical formalism entails is a non-Boolean type of logic. Let's see why this is the case.

The postulates of quantum mechanics are the following.

1. The state space of a system S is represented by the Hilbert space $\mathcal{H}$ (vector space) such that each element of $\mathcal{H}$ represents a possible configuration of the system.

2. Any physical quantity A is represented by self adjoint operator $\hat{A}$ which acts on the states in $\mathcal{H}$

3. Propositions are represented by particular linear subspace of the Hilbert space $\mathcal{H}$ i.e. propositions are identified as projection operators $\hat{P}$ onto the vector space $\mathcal{H}$ such that the subspace of $\mathcal{H}$ into which they project,

defined as $\mathcal{H}_{\hat{P}} = \{\vec{\Psi} \in \mathcal{H} | \hat{P}\vec{\Psi} = \vec{\Psi}\}$, represents the subspace of $\mathcal{H}$ in which these propositions are true. Therefore we can identify propositions as linear subspaces of $\mathcal{H}$.

Since postulate number 3 is central to this paper it is worth analysing in more details how it is derived from the formalism of Quantum Mechanics. The identification of propositions with projector operators is a consequence of a postulate and two theorems of Quantum Mechanics, namely:

- Projection operators $\hat{P}$ satisfies the following relation

$$P = P^2$$

  which means that $\hat{P}$ has only two eigenvalues 1 or 0, such that eigenvalue 1 has associated eigenvector $|1\rangle \in \mathcal{H}_{\hat{P}}$, while 0 has associated eigenvector $|0\rangle \in \mathcal{H}_{\hat{P}}{}^{\perp}$.

- **Theorem 3.1** *(Spectral Theorem) [3], [15], [7]. Suppose $\hat{A}$ is a compact self-adjoint operator on a Hilbert space $\mathcal{H}$ then there is an orthonormal basis of $\mathcal{H}$ consisting of eigenvectors of $\hat{A}$. Each eigenvalue of $\hat{A}$ is real.*

  Since the set of all eigenvectors of a self adjoint operator $\hat{A}$ (which represents the physical quantity A) is an orthonormal basis set for $\mathcal{H}$, it follows that any vector $\vec{\Psi}$ can be expanded in the following way:

$$\vec{\Psi} = \sum_{m=1}^{M} \sum_{j=1}^{d(m)} \Psi_{mj} \vec{v}_{mj}$$

  where d(m) is the dimension of the linear subspace of $\mathcal{H}$ created by the set of eigenvectors with eigenvalue a_m. In Dirac notation the above expansion theorem is :

$$\vec{\Psi} = \sum_{m=1}^{M} \sum_{j=1}^{d(m)} \langle a_m, j | \Psi \rangle | a_m, j \rangle$$

As a consequence any operator $\hat{A}$ can be written in terms of its spectral representation

$$\hat{A} = \sum_{m=1}^{M} a_m \hat{P}_m$$

where

$$\hat{P} = \sum_{j=1}^{d(m)} |a_m, j\rangle\langle a_m, j|$$

is the spectral projector of $\hat{A}$ that projects onto the subspace of eigenvectors of $\hat{A}$ with eigenvalue (a)

Proof 3.1

$$\begin{aligned}
\hat{A}|\Psi\rangle &= \sum_{m=1}^{M}\sum_{j=1}^{d(m)} \hat{A}\Psi|a_m, j\rangle \\
&= \sum_{m=1}^{M}\sum_{j=1}^{d(m)} a_m\langle a_m, j|\Psi\rangle|a_m, j\rangle \\
&= \sum_{m=1}^{M}\sum_{j=1}^{d(m)} a_m, j|a_m\rangle\langle a_m, j|\Psi\rangle \\
&= \sum_{m=1}^{M}\sum_{j=1}^{d(m)} a_m \hat{P}_m^j|\Psi\rangle \\
&= \sum_{m=1}^{M} a_m \hat{P}_m|\Psi\rangle \\
\therefore \quad \hat{A} &= \sum_{m=1}^{M} a_m \hat{P}_m.
\end{aligned}$$

Generally we have

$$\hat{P}_{A\in\Delta} = \sum_{a\in\Delta} \hat{P}_{A=a}$$

that projects onto the subspace of eigenvectors of $\hat{A}$ whose eigenvalues lie within the subset $\Delta \subset \mathbb{R}$.

- **Theorem 3.2** *1. the only possible result of a measurement of an observable represented by self adjoint operator $\hat{A}$ is an eigenvalue of $\hat{A}$*

2. *If a measurement of the quantity A is made on a system in state* $|\Psi\rangle$ *then the probability of obtaining as a result an eigenvalue a of* $\hat{A}$ *is*

$$Prob(A = a_n, |\Psi\rangle) = \langle\Psi|\hat{P}_n|\Psi\rangle$$

As a consequence of this theorem the spectral projector $\hat{P}$ represents an observable that can have as values only 1 and 0 since $P = P^2$. This implies that $\hat{P}$, represents propositions of the form : " if A is measured, the result of the measurement will lie in the subset $A \subset \mathbb{R}$ of $\mathbb{R}$ ".

From the above we can deduce that propositions $(A \in \Delta)$ are represented by spectral projectors $\hat{E}[A \in \Delta] = \hat{P}_{A\in\Delta}$. Therefore propositions are identified with subspaces $\mathcal{H}_{\hat{P}} = \{\vec{\Psi} \in \mathcal{H} | \hat{P}\vec{\Psi} = \vec{\Psi}\}$ of the Hilbert space $\mathcal{H}$ for which they are true.

Similarly to classical physics, the quantum mechanics projection operators, can represent more than one proposition, therefore, each linear subspace of $\mathcal{H}$ corresponds to equivalent classes of propositions. this enables us to define the logical connectives in terms of operations on the relevant subspaces of $\mathcal{H}$.

Therefore, given propositions X_i's and subspaces of $\mathcal{H}$ W_{X_i}'s we have:

$$X_1 \vee X_2 \longleftrightarrow W_{X_1} + W_{X_2}$$

$$X_1 \wedge X_2 \longleftrightarrow W_{X_1} \cap W_{X_2}$$

$$X_1 \Longrightarrow X_2 \longleftrightarrow W_{X_1} \subseteq W_{X_2}$$

$$X_1 \Longleftrightarrow X_2 \longleftrightarrow W_{X_1} = W_{X_2}$$

$$\neg X \longleftrightarrow W_{\neg X} = W_X^{\perp}$$

$$1 \longleftrightarrow \mathcal{H}$$

$$0 \longleftrightarrow 0$$

where $W_X^{\perp}$ represents the orthogonal subspace of W_X in $\mathcal{H}$, and $W_{X_1} + W_{X_2}$ is the linear sum of the subspaces W_{X_1} and W_{X_2}.

The fact that $X_1 \vee X_2$ corresponds to $W_{X_1} + W_{X_2}$ instead of $W_1 \cup W_2$ (which is not a linear subspace of $\mathcal{H}$[1]) is one of the central differences between classical mechanics and quantum mechanics [2]. The above definitions for logical connectives entail that Quantum Logic is non distributive i.e.

$$W_{X_1} \cap (W_{X_2} + W_{X_3}) \neq (W_{X_1} \cap W_{X_2}) + (W_{X_1} \cap W_{X_3})$$

therefore, quantum logic seems to be not Boolean.

We could also write the above logical connectives in terms of the spectral projectors $\hat{P}$, in fact the set of all such projectors in a Hilbert space $\mathcal{H}$ forms a lattice $P(\mathcal{H})$ under subset inclusion of those subsets $W_i \subseteq \mathcal{H}$ onto which the projectors $\hat{P}$ project, i.e.

$$\hat{P}_1 \leq \hat{P}_2 \qquad \longleftrightarrow \qquad W_{\hat{P}_2} \subseteq W_{\hat{P}_1} \qquad \longleftrightarrow \qquad \hat{P}_1 \cdot \hat{P}_2 = \hat{P}_1 \cdot \hat{P}_2 = \hat{P}_1$$

This means that the projector $\hat{P}_1$ projects onto a subspace of $\mathcal{H}$ which, in turn, is a subset of the space onto which projector $\hat{P}_2$ projects. Expressing the logical

[1] $W_1 \cup W_2$ fails to be a linear subspace of $\mathcal{H}$ since it is not closed with respect to the sum operation between vectors

[2] Note that $X_1 \vee X_2$, in order to account for the cases in which $\mathcal{H}$ is infinite dimension, should really be written as $\overline{W_{X_1} + W_{X_2}}$, which represents the closure of $W_{X_1} + W_{X_2}$ i.e. "the smallest topologically closed linear subspace of $\mathcal{H}$ which contains W_{X_1} and W_{X_2}" [4]

connectives in terms of the relations between the spectral projectors $\hat{P}$ we have:

$$X_1 \wedge X_2 \longleftrightarrow \hat{P}_{X_1} \wedge \hat{P}_{X_2}$$

$$X_1 \vee X_2 \longleftrightarrow \hat{P}_{X_1} \vee \hat{P}_{X_2} = (\hat{P}_{X_1} + \hat{P}_{X_2} - \hat{P}_{X_1}\hat{P}_{X_2})$$

$$X_1 \Rightarrow X_2 \longleftrightarrow \hat{P}_{X_1} \leq \hat{P}_{X_2}$$

$$X_1 \Leftrightarrow X_2 \longleftrightarrow \hat{P}_{X_1} = \hat{P}_{X_2}$$

$$\neg X \longleftrightarrow \hat{P}_{\neg X} = \hat{1} - \hat{P}_X$$

$$1 \longleftrightarrow \hat{1}$$

$$0 \longleftrightarrow \hat{0}$$

An important distinction between classical logic and quantum logic that is worth mentioning is the following: in classical logic, if a proposition is true, then, the negation of that proposition is false. In Quantum Mechanics this does not hold. To illustrate, let us consider a vector $\vec{\Psi} \in \mathcal{H}$. Given a subspace W of $\mathcal{H}$, $\vec{\Psi}$ can have non zero components in both W and $W^{\perp}$. Since a proposition $A \in \Delta$ is said to be true in a state $\vec{\Psi}$ if $\vec{\Psi} \in W$ where $W = \mathcal{H}_{A\in\Delta}$ (subset of the Hilbert space $\mathcal{H}$ in which the proposition $A \in \Delta$ is true) and false if $\vec{\Psi} \notin W$, then if $\vec{\Psi}$ had non zero components in both W and $W^{\perp}$ the proposition $A \in \Delta$ could not be assessed to be either true or false. This suggests that even if a proposition is evaluated as being false, this does not entail that the negation of that proposition should be evaluated as true.

3.1 Lattices in Quantum Logic

As stated in the previous section, the structure of Quantum Logic is derived from the correspondence of propositions with projection operators. Since Projectors are associated with closed subspaces of $\mathcal{H}$, the mathematical structure of Quantum Logic can be derived from the mathematical structure of the set of closed subspaces of $\mathcal{H}$. It then follows that the binary operations of "meet" and "join" can be defined in terms of operations between subsets of $\mathcal{H}$, therefore Quantum propositions form a lattice. As will come clear later on, the lattice of Quantum propositions is an *orthocomplemented* but not *distributive* lattice.

An essential ingredient in understanding how lattices of propositions are built in Quantum Mechanics is the fact that in Quantum Mechanics each dynamical variable can be written as a sum (or integral) of projection operators on the Hilbert space $\mathcal{H}$ i.e.

$$\hat{A} = \sum_{m=1}^{M} a_m \hat{P}_m$$

such that commuting projectors are represented by orthogonal subspaces in the Hilbert space $\mathcal{H}$.

In what follows I will quote an example given by Jeffrey Bub in his book "Interpreting the Quantum World", where he considers a 2 dimensional Hilbert space $\mathcal{H}_2$ and two quantities A and B representing spin in two different directions

$$\hat{A} = a_1 \hat{P}_{a_1} + a_2 \hat{P}_{a_2}$$

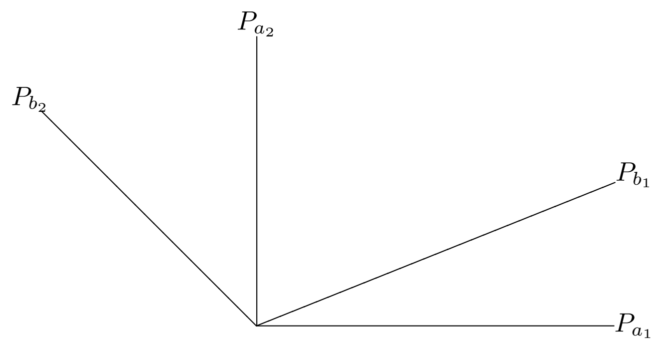
$$\hat{B} = b_1 \hat{P}_{b_1} + b_2 \hat{P}_{b_2}$$

In this example the $\hat{P}$'s represents $\pm\frac{1}{2}$ values of the spin, and $\hat{P}_{a_1}$ and $\hat{P}_{a_2}$ are projection operators that project onto the orthogonal pair of one dimensional eigenspaces of $\hat{A}$, i.e. they belong to the spectral decomposition of $\hat{A}$.

What I said above is also valid for the projectors in the spectra decomposition of B.

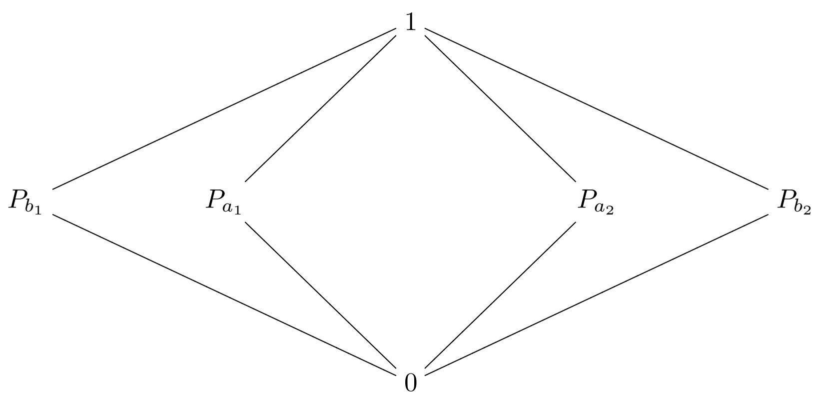
Graphically we have [10]:

Diagram 3.1



with respective lattice of properties $P(\mathcal{H}_2)$ [10]

Diagram 3.2



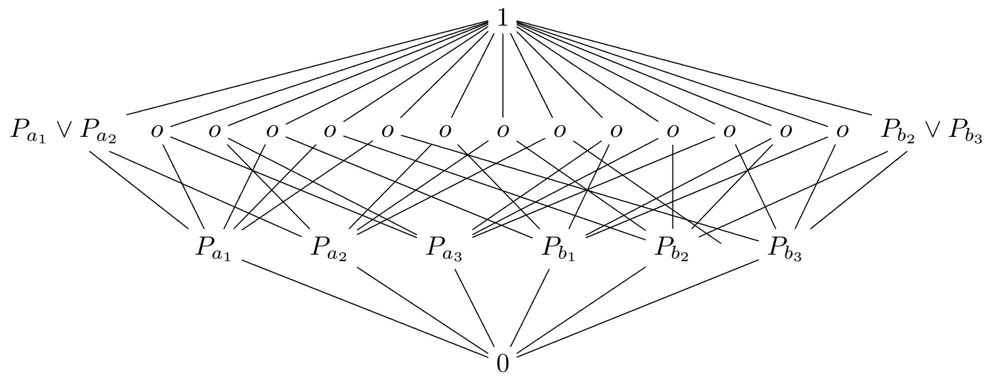
Instead, if we had a 3 dimensional Hilbert space $\mathcal{H}_3$ such that

$$\hat{A} = a_1 \hat{P}_{a_1} + a_2 \hat{P}_{a_2} + a_3 \hat{P}_{a_3}$$

$$\hat{B} = b_1 \hat{P}_{b_1} + b_2 \hat{P}_{b_2} + b_3 \hat{P}_{b_3}$$

the lattice of properties $P(\mathcal{H}_3)$ would be [10]:

Diagram 3.3



here the $\hat{P}_{a_i}$ are orthogonal between each other and the $\hat{P}_{b_i}$ are orthogonal between each other.

Each sublattice of $P(\mathcal{H}_3)$, composed by all the statements which refer to commuting pairs of projection operators, is isomorphic to a Boolean lattice i.e. the sublattice formed by $\hat{0}$,$\hat{1}$,$\hat{P}_{a_1}$,$\hat{P}_{a_2}$,$\hat{P}_{a_3}$,$\hat{P}_{a_1} \vee \hat{P}_{a_2}$ $\hat{P}_{a_1} \vee \hat{P}_{a_3}$,$\hat{P}_{a_2} \vee \hat{P}_{a_3}$ is a Boolean sublattice of $\mathcal{H}_3$. Generally Boolean sublattices of $P(\mathcal{H})$ are formed by a set of orthogonal vectors that span $\mathcal{H}$.

The relations that exist between spectral projectors associated to different self-adjoint operators are determined by the relative spectral algebras.

3.1.1 Spectral Algebras

Definition 3.1 *:A* ***spectral algebra*** W_A *of an operator* A *is the Boolean algebra associated with those projector that form the spectral decomposition of* A*, projectors that project onto the eigenspaces associated with the eigenvectors of* A*.*

In order to explain the above definition, and the existing relations between different spectral algebras associated with different self-adjoint operators, we will consider another example of lattice of properties for a quantum system given by Bub in his book “Interpreting the Quantum World”.

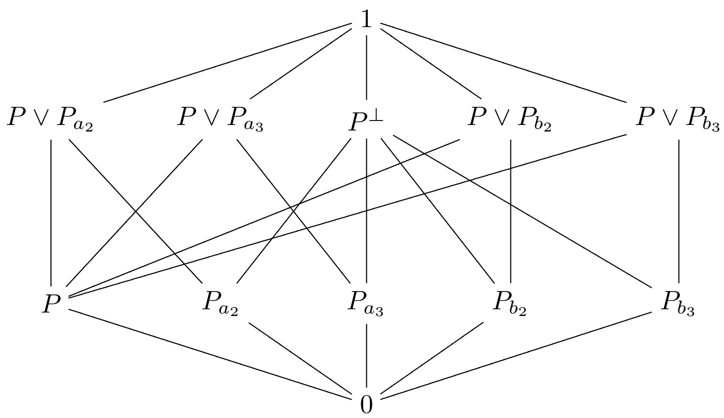
If we consider two physical quantities A and B, whose representative operators have the following spectral decomposition:

$$\hat{A} = a_1\hat{P} + a_2\hat{P}_{a_2} + a_3\hat{P}_{a_3} \tag{3.1}$$

$$\hat{B} = b_1\hat{P} + b_2\hat{P}_{b_2} + b_3\hat{P}_{b_3} \tag{3.2}$$

such that the operators $\hat{A}$ and $\hat{B}$ have a common projector $\hat{P}$, the lattice of properties would then be [10]

Diagram 3.4



where

$$\hat{P}^{\perp} = \hat{P}_{a_2} \vee \hat{P}_{a_3} = \hat{P}_{b_1} \vee \hat{P}_{b_3} = \hat{P}_{a_2} \vee \hat{P}_{b_2} = \hat{P}_{a_2} \vee \hat{P}_{b_3} = \hat{P}_{a_3} \vee \hat{P}_{b_2} = \hat{P}_{a_3} \vee \hat{P}_{b_3}$$

Now consider a Borel function $f : \sigma(\hat{A}) \to \mathbb{R}$ such that $\hat{B} = f(\hat{A})$.

According to diagram 3.4 the spectral projectors of $\hat{A}$ are $\hat{P}, \hat{P}_{a_2}, \hat{P}_{a_3}$ with respective eigenvalues a_1, a_2, a_3,. Therefore, the Boolean sublattice of $P(\mathcal{H})$, which represents the spectral algebra W_A of $\hat{A}$ is composed by $\hat{0}$, $\hat{1}$, $\hat{P}^{\perp}$, $\hat{P}$,$\hat{P}_{a_2}$, and $\hat{P}_{a_3}$. If we choose f such that the eigenvalues of $\hat{B}$ are $f(a_1), f(a_2) = f(a_3)$, then the spectral projectors of $\hat{B}$ are $\hat{P}$ and $\hat{P}^{\perp}$. This entails that the spectral algebra $W_{f(A)}$ of the operator $\hat{B}$ is a subalgebra of W_A, i.e. $W_{f(A)} \subseteq W_A$. Since in Quantum Mechanics propositions are represented by spectral projectors the question that arises is the following: how do the relations between different spectral projectors determine the relations between the propositions that they represent? To answer this question let us again consider the above example. In this example the proposition "$A = a_1$" is represented by the projection operator $\hat{P}_{A=a_1}$ which is equivalent to $\hat{P}$ which, in turn, is equivalent to $\hat{P}_{f(A)=f(a_1)}$ since f is one to one with respect to a_1, therefore, in this case, the two propositions "$A = a_1$" and "$f(A) = f(a_1)$" are equivalent. If, instead, we consider propositions "$A = a_2$" and "$f(A) = f(a_2)$, then, the situation changes since f is many to one with respect to a_2. In fact, we would have the following: $\hat{P}_{f(A)=f(a_2)} = \hat{P}^{\perp} = \hat{P}_{a_2} \vee \hat{P}_{a_3} = \hat{P}_{A=a_2} \vee \hat{P}_{A=a_3}$. Since $\hat{P}_{a_2} \vee \hat{P}_{a_3}$ represents the join of $\hat{P}_{a_2}$ and $\hat{P}_{a_3}$ it follows, by definition of join (Definition 2.2) , that $\hat{P}_{A=a_2} \leq \hat{P}_{f(A)=f(a_2)}$, with respect to the partial ordering of the lattice. Therefore the proposition "$f(A) = f(a_2)$" is weaker than proposition "$A = a_2$" and so we say that "$f(A) = f(a_2)$" is the coarse graining of "$A = a_2$". (We will return on this later).

3.1.2 Valuating Propositions

As previously stated, in the classical case, truth values are assigned through a homeomorphism from the Boolean lattice of projectors to be evaluated, to the Boolean lattice $\{0, 1\}$ which is equivalent to the lattice $P(\mathcal{H}_1)$ of the 1 dimension Hilbert space, i.e.

Diagram 3.5

Unfortunately in the quantum case, things are not that simple, since no homeomorphism exists between non Boolean lattices such that the inner structure of the lattices is preserved. In fact if a homeomorphism $h : P(\mathcal{H}_3) \rightarrow P(\mathcal{H}_1)$ existed between the lattice $P(\mathcal{H}_3)$ and $(P\mathcal{H}_1)$, then h would map one and only one projector of each orthogonal pair onto $\hat{1}$, while the remaining projector would be mapped onto $\hat{0}$ such that their conjunction in $P(\mathcal{H}_1)$ would give $\hat{0}$. This, though, would violate the orthogonality condition since orthogonal pairs in $P(\mathcal{H}_3)$ would be mapped onto $\hat{0}$. Moreover if non orthogonal rays are mapped onto $\hat{1}$ in $P(\mathcal{H}_1)$ then their intersection in $P(\mathcal{H}_1)$ would be $\hat{1}$, while their intersection in $P(\mathcal{H}_3)$ would be $\hat{0}$ therefore the homeomorphism h would not be structure preserving. Extending this argument it can be proven that not even homeomorphism between Boolean sublattices of $P(\mathcal{H}_3)$ and $P(\mathcal{H}_1)$ do exist, except for the homeomorphism $h : P(\mathcal{H}_2) \rightarrow P(\mathcal{H}_1)$, in this case one projector from each pair of orthogonal projectors would be mapped onto $\hat{1}$ while the other would be mapped onto $\hat{0}$.

What all this means is that it is impossible to assign truth values belonging to the Boolean algebra $\{0, 1\}$, to subspaces of the Hilbert space $\mathcal{H}$ (which represents propositions) when $dim\mathcal{H} > 2$, in such a way that the structure of the Boolean sublattices of $\mathcal{H}$ is respected. This is one of the many versions of the Kochen-Specker theorem.

Chapter 4

The Kochen-Specker Theorem

The Kochen-Specker Theorem derives from the incompatibility of two assumptions regarding observables in Quantum Mechanics, namely [1].

1. The need of assigning simultaneous truth values to all observables associated with projectors in $\mathcal{O}$.

2. The need for the values of observables to be "mutually exclusively and collectively exhaustable" [10].

From the above statement it follows that the Kochen-Specker Theorem is related to the existence, in Quantum Mechanics, of a value function $V_{\vec{\Psi}} : \mathcal{O} \rightarrow \mathbb{R}$ from the set of self-adjoint operators to the Reals.

$V_{\vec{\Psi}}$ assigns to each quantity A, which is represented by the self-adjoint operator $\hat{A} \in \mathcal{O}$, a real number that represents the value of A for each state $\vec{\Psi}$ of the system. A natural condition a value function should satisfy is the :

Functional Composition Condition (FUNC):

for any function $f : \mathbb{R} \to \mathbb{R}$, the following condition is satisfied

$$V_{\vec{\Psi}}(f(\hat{A})) = V_{\vec{\Psi}}(\widehat{f(A)}) = f(V_{\vec{\Psi}}(\hat{A})) \tag{4.1}$$

where $\widehat{f(A)}$ represents the operator defined by $f(\hat{A})$. The condition FUNC entails the following:

1. *sum rule*

$$V_{\vec{\Psi}}(\hat{A}+\hat{B}) = V_{\vec{\Psi}}(\hat{A}) + V_{\vec{\Psi}}(\hat{B}) \tag{4.2}$$

   where $\hat{A}$ and $\hat{B}$ are such that $[\hat{A}\hat{B}] = 0$

   Proof 4.1 *to prove the above result we need the following theorem:*

   Theorem 4.1 *: given a pair of self adjoint operators $\hat{A}$ and $\hat{B}$ such that $[\hat{A}\hat{B}] = 0$ and two functions $f, g : \mathbb{R} \to \mathbb{R}$ then, there exists a third operator $\hat{C}$ such that $\hat{A} = f(\hat{C})$ and $\hat{B} = g(\hat{C})$.*

   Given two commuting operators $\hat{A}$ and $\hat{B}$ from the above theorem it follows that $\hat{A} = f(\hat{C})$ and $\hat{B} = g(\hat{C})$, therefore there exists a function h=f o g such that $\hat{A}+\hat{B} = h(\hat{C})$ therefore

$$\begin{aligned} V_{\vec{\Psi}}(\hat{A}+\hat{B}) &= V_{\vec{\Psi}}(h(\hat{C})) \\ &= h(V_{\vec{\Psi}}(\hat{C})) \\ &= f(V_{\vec{\Psi}}(\hat{C})) + g(V_{\vec{\Psi}}(\hat{C})) \\ &= V_{\vec{\Psi}}(f(\hat{C})) + V_{\vec{\Psi}}(g(\hat{C})) \\ &= V_{\vec{\Psi}}(\hat{A}) + V_{\vec{\Psi}}(\hat{B}) \end{aligned}$$

2. *product rule*

$$V_{\vec{\Psi}}(\hat{A}\hat{B}) = V_{\vec{\Psi}}(\hat{A})V_{\vec{\Psi}}(\hat{B}) \tag{4.3}$$

where $\hat{A}$ and $\hat{B}$ are such that $[\hat{A}\hat{B}] = 0$

Proof 4.2 *Given* $\hat{A} = f(\hat{C})$ *and* $\hat{B} = g(\hat{C})$ *there exists a function* $k = f \cdot g$ *such that* $\hat{A} \cdot \hat{B} = k(\hat{C})$ *therefore*

$$\begin{aligned} V_{\vec{\Psi}}(\hat{A} \cdot \hat{B}) &= V_{\vec{\Psi}}(k(\hat{C})) \\ &= k(V_{\vec{\Psi}}(\hat{C})) \\ &= f(V_{\vec{\Psi}}(\hat{C})) \cdot g(V_{\vec{\Psi}}(\hat{C})) \\ &= V_{\vec{\Psi}}(f(\hat{C})) \cdot V_{\vec{\Psi}}(g(\hat{C})) \\ &= V_{\vec{\Psi}}(\hat{A}) \cdot V_{\vec{\Psi}}(\hat{B}) \end{aligned}$$

As a consequence of the product and sum rules we obtain the following equalities:

$$\begin{aligned} V_{\vec{\Psi}}(\hat{1}) &= 1 \\ V_{\vec{\Psi}}(\hat{0}) &= 0 \\ V_{\vec{\Psi}}(\hat{P}) &= 0 \quad or \quad 1 \end{aligned} \tag{4.4}$$

Since the set of all eigenvectors of a self adjoint operator $\hat{A}$ forms an orthonormal basis for $\mathcal{H}$, then we have the following resolution of unity

$$\hat{\mathbb{1}} = \sum_{m=1}^{M} \hat{P}_m \tag{4.5}$$

From 4.2, 4.3, 4.4 and 4.5 we conclude (for discreet case but it can easily be extended to the continuous case)

$$V_{\vec{\Psi}}(\hat{\mathbb{1}}) = V_{\vec{\Psi}}\left(\sum_{m=1}^{M} \hat{P}_m\right) = \sum_{m=1}^{M} V_{\vec{\Psi}}(\hat{P}_m) = 1 \tag{4.6}$$

What this equation means is that one and only one of the projectors that form the resolution of unity gets assigned the value 1 (true), while the rest gets assigned the value 0 (false), i.e. the value assignment is said to be “ mutually exclusive and collectively exhaustive" [10]. As said before, however, it is impossible to give simultaneous values to all observables associated with a set of self adjoint operators, in such a way that, the values are "mutually exclusive and collectively exhaustive". Since the property of values of being "mutually exclusive and collectively exhaustive" is a consequence of FUNC, it is worth analysing how the condition FUNC is derived from the formalism of Quantum Mechanics.

FUNC is a direct consequence of three assumptions and a principle derived from the formalism of Quantum Mechanics [6]:

- **Statistical functional compositional principle**:*given a self adjoint operator $\hat{A}$ that represents an observable A, a function $f : \mathbb{R} \rightarrow \mathbb{R}$ and a state $\vec{\Psi}$, then for an arbitrary real number a we have the following equality:*

  $$prob[V_{\vec{\Psi}}(f(A)) = a] = prob[f(V_{\vec{\Psi}}(A)) = a]$$

  In order to prove the above principle we have to define the relation between projector operators and their respective characteristic functions.

  Let us consider the following characteristic function (Definition 8.11) χ_r such that

  $$\chi_r(X) = \begin{cases} 1 & if \quad X = r \\ 0 & otherwise \end{cases}$$

  It then follows that, given a self adjoint operator $\hat{A}$ whose spectral decomposition contains the spectral projector $\hat{P}_m$ one can write:

  $$\chi_r(\hat{A}) := \sum_{m=1}^{M} \chi_r a_m \hat{P}_m = \begin{cases} \hat{P}_m & if \quad a_m = r \\ 0 & otherwise \end{cases} \tag{4.7}$$

What equation 4.7 uncovers is that $\chi_r(\hat{A}) = \hat{P}_m$ iff r is an eigenvalue of $\hat{A}$. Moreover, given a function $f : \sigma(\hat{A}) \rightarrow \mathbb{R}$ we have:

$$\chi_r(f(\hat{A})) = \chi_{f^{-1}(a)}(\hat{A}) \tag{4.8}$$

We know that the statistical algorithm [3] for projection operators is

$$prob(V_{\vec{\Psi}}((\hat{A}) = a_m) = Tr(\hat{P}_m \cdot \hat{P}_{\vec{\Psi}}) \tag{4.9}$$

which means that if a measurement of an observable A is made on a system in state $\vec{\Psi}$, then, the probability of obtaining as a result the eigenvalue a_m is given by 4.9.

Therefore from 4.7 and 4.9 we get

$$prob(V_{\vec{\Psi}}(\hat{A}) = a_m) = Tr(\chi_{a_m}(\hat{A}) \cdot \hat{P}_{\vec{\Psi}})$$

We can now prove the Statistical functional compositional principle [6].

Proof 4.3 *Using equations:* 4.7 ,4.9 *and* 4.8 *we can write the statistical algorithm for projector operators as follows*

$$\begin{aligned} prob(V_{\vec{\Psi}}(f(A)) = b) &= Tr((\chi_{f^{-1}(b)}(\hat{A}) \cdot \hat{P}_{\vec{\Psi}}) \\ &= Tr(\hat{P}_{f^{-1}(b)} \cdot \hat{P}_{\vec{\Psi}}) \\ &= prob(V_{\vec{\Psi}}(A) = f^{-1}(b)) \end{aligned}$$

but

$$V(A) = f^{-1}(b) \quad \Leftrightarrow \quad f(V(A)) = b$$

therefore

$$prob(V_{\vec{\Psi}}(f(A)) = b) = prob(f(V_{\vec{\Psi}}(A)) = b)$$

- **Noncontextuality**: the value of observables is independent of the measurement context, i.e. the value of each observable is independent of any other observables evaluated at the same time.
- **Value definiteness**: observables posses definite values at all times.
- **Value realism**: to each real number α such that $\alpha = prob(V(\hat{A}) = \beta)$ for an operator $\hat{A}$ there corresponds an observable A with value β.

From the above conditions (1),(2),(3)and (4) the FUNC condition follows.

Proof 4.4 *Consider an observable B.*

From (3) we deduce that B possesses a value: V(B)=b.

Given a function $f : \mathbb{R} \to \mathbb{R}$ *we obtain the quantity f(V(B))=a.*

Applying (1) we get prob[f(V(B))=a]=prob[V(f(B))=a] which means that there exist a self adjoint operator of the form $f(\hat{B})$*.*

From (4) it then follows that the corresponding observable for $f(\hat{B})$ *has value a, therefore f(V(B))=V(f(B)).*

From (2) this result is unique, therefore FUNC follows.

We now state the Kochen-Specker Theorem.

Theorem 4.2 ***Kochen-Specker Theorem****: if the dimension of* $\mathcal{H}$ *is greater than 2, then, there, does not exist any valuation function* $V_{\vec{\Psi}} : O \to \mathbb{R}$ *from the set* $\mathcal{O}$ *of all bounded self adjoint operators* $\hat{A}$ *of* $\mathcal{H}$ *to the reals* $\mathbb{R}$ *such that, the functional composition principle is satisfied for all* $\hat{A} \in \mathcal{O}$ *and all* $\vec{\Psi} \in \mathcal{H}$*.*

Another way of stating the theorem which seems more useful for developing a proof is the following

Theorem 4.3 ***Kochen-Specker Theorem****: given a Hilbert space* $\mathcal{H}$ *such that*

$dim\mathcal{H} > 2$ and a set $\mathcal{O}$ of self adjoint operators $\hat{A}$ which represent observables, then the following two statements are contradictory:

1. *all observables associated with projectors in $\mathcal{O}$ have values simultaneously i.e. they are mapped uniquely onto the reals.*

2. *the values of observables follow the functional composition principle (FUNC).*

4.1 Proof of Kochen-Specker Theorem

There are various proofs of the Kochen-Specker Theorem.We hereafter describe a simplified version of the proof due to Kernaghan (1994) [12].

In his proof, Kernaghan considers a real 4 dimensional Hilbert space $\mathcal{H}_4$ (there is no loss in generality in considering the Hilbert space to be real). According to this proof the Kochen-Specker theorem reduces to the coloring in problem, i.e. "within every set of orthogonal vectors in $\mathcal{H}_4$ exactly one must be colored white (1,true) while the remaining black (0,false)". If we write down all the various possibilities for a set of 20 vectors, we would end up with the following table where each column denotes a set of 4 orthogonal vectors:

1,0,0,0	1,0,0,0	1,0,0,0	1,0,0,0	-1,1,1,1	-1,1,1,1	1,-1,1,1	1,1,-1,1	0,1,-1,0	0,0,1,-1	1,0,1,0
0,1,0,0	0,1,0,0	0,0,1,0	0,0,0,1	1,-1,1,1	1,1,-1,1	1,1,-1,1	1,1,1,-1	1,0,0,-1	1,-1,0,0	0,1,0,1
0,0,1,0	0,0,1,1	0,1,0,1	0,1,1,0	1,1,-1,1	1,0,1,0	0,1,1,0	0,0,1,1	1,1,1,1	1,1,1,1	1,1,-1,-1
0,0,0,1	0,0,1,-1	0,1,0,-1	0,1,-1,0	1,1,1,-1	0,1,0,-1	1,0,0,-1	1,-1,0,0	1,-1,-1,1	1,1,-1,-1	1,-1,-1,1

It is easy to see from the table that condition 4.6 is not satisfied. In fact if it were satisfied we would end up with 11 entries being colored white, since each column would have exactly one entry that is colored white and there are 11 columns. But from the above table we can see that the total number of white entries is greater than 11 (given a non-contextual assignment of the entries i.e. we are assuming that same vectors get assigned same color independently of the column they belong to). Therefore we conclude that it is impossible to obtain a coloring of a set of orthogonal vectors that is consistent with condition 4.6. Although this is a very simplified version of the proof of the Kochen-Specker theorem, the main idea is the same as the main idea in the original proof namely: given a set of orthogonal vectors in $\mathcal{H}$ it is impossible to assign to each of them a set of numbers $\{1, 0, 0, 0....0\}$ where only one entry is equal to 1, i.e.it is impossible to give simultaneous values to all observables while respecting the FUNC condition. In the remaining of this paper we will describe a way of overcoming the no-go Kochen-Specker Theorem

4.2 "Solution" to the Kochen Specker Theorem

How can the Kochen-Specker Theorem be redefined so that Quantum Theory can still be given the status of a realist theory?

As we have, seen the Kochen-Specker Theorem asserts that, either we abandon the set $\{0, 1\}$ as the possible truth values, adopting instead some kind of multi-valued logic, or we abandon the functional composition principle as we know it. Abandoning FUNC would entail abandoning some or all of the three assumptions:

1. *Non-contextuality*
2. *Value Definiteness*
3. *Value Realism*

from which it derives. In what follows I will try to show that the negation of the first two assumptions above is a direct consequence of the mathematical formalism of Quantum Theory.

1. *Quantum Mechanics is contextual*

   In order to prove this, we need to define what is meant for two observables to be simultaneously measurable.

   Definition 4.1 *Given two observables $\hat{A}$ and $\hat{B}$ we say that they are **simultaneously measurable** iff $[\hat{A}\hat{B}] = 0$.*

   In terms of lattices of spectral operators, the above condition can be written as follows.

   Definition 4.2 *two observables $\hat{A}$ and $\hat{B}$ in $\mathcal{H}_n$ are **simultaneously measurable** iff their spectral algebras are embedded in the same Boolean sublattice of $(P(\mathcal{H}_n))$.*

   Let us now consider three observables: $\hat{A}$, $\hat{B}$, $\hat{C}$ such that $[\hat{A}\hat{C}] = 0 = [\hat{B}\hat{C}]$ but $[\hat{A}\hat{B}] \neq 0$. If $\hat{A}$ and $\hat{B}$ are such that their spectral decomposition contains a common projector $\hat{P}$ as in diagram 3.4, whose spectral decomposition is expressed by equation 3.1 and equation 3.2, then, from equation 4.7 it follows that $\hat{P}$ can be expressed in terms of $\hat{A}$ or of $\hat{B}$, i.e.:

   $$\hat{P} = \chi_{a_1}(\hat{A})$$

   $$\hat{P} = \chi_{b_1}(\hat{B})$$

   Since commuting operators correspond to orthogonal operators, then, if we choose to express $\hat{P}$ in terms of $\hat{A}$ i.e. $\hat{P} = \chi_{a_1}(\hat{A})$ the commuting operators of $\hat{P}$ are: $\hat{P}_{a_2}$, $\hat{P}_{a_3}$,$\hat{P} \vee \hat{P}_{a_2}$,$\hat{P} \vee \hat{P}_{a_3}$ and $\hat{P}^{\perp}$. If, instead, we choose to express $\hat{P}$ in terms of $\hat{B}$ i.e. $\hat{P} = \chi_{b_1}(\hat{B})$ then, the commuting operators of $\hat{P}$ would be: $\hat{P}_{b_2}$, $\hat{P}_{b_3}$,$\hat{P} \vee \hat{P}_{b_2}$,$\hat{P} \vee \hat{P}_{b_3}$ and $\hat{P}^{\perp}$. Therefore the value of $\hat{P}$ would be

$$\begin{aligned} \hat{P} = \chi_{a_1}(\hat{A}) \quad \Longleftrightarrow \quad & [\hat{P}\hat{P}_{a_2}] = \\ & = [\hat{P}\hat{P}_{a_3}] \\ & = [\hat{P}\hat{P} \vee \hat{P}_{a_2}] \\ & = [\hat{P}\hat{P} \vee \hat{P}_{a_3}] \\ & = [\hat{P}\hat{P}^{\perp}] \\ & = 0 \end{aligned}$$

or it would be

$$\begin{aligned} \hat{P} = \chi_{b_1}(\hat{B}) \quad \Longleftrightarrow \quad & [\hat{P}\hat{P}_{b_2}] = \\ & = [\hat{P}\hat{P}_{b_3}] \\ & = [\hat{P}\hat{P} \vee \hat{P}_{b_2}] \\ & = [\hat{P}\hat{P} \vee \hat{P}_{b_3}] \\ & = [\hat{P}\hat{P}^{\perp}] \\ & = 0 \end{aligned}$$

therefore, the value of $\hat{P}$ is contextual.

2. *Observables do not have definite values at all times.*

   Let us consider a two dimensional Hilbert space $\mathcal{H}_2$ and a system with state vector:

$$|\Psi\rangle = \frac{1}{2}(|\uparrow\rangle - |\downarrow\rangle) \tag{4.10}$$

$|\uparrow\rangle$ and $|\downarrow\rangle$ represent the spin up and spin down in the x-direction, therefore, the state represented by 4.10 is a superposition of eigenstates of $\hat{S}_x$. In this situation it is meaningless to ask whether the particle has spin up or spin down; all that can be asserted with certainty is the value of the probabilities of obtaining spin up or spin down as a result of a measurement. Therefore it is inconsistent to say that the particle possesses the property of being spin up or spin down.

If properties in Quantum Mechanics are contextual, it follows that truth values should be contextual as well. As mentioned in Chapter 3, there are propositions in Quantum Mechanics that can not be assessed to be either true or false, leading in such a way to two types of "false": one referring to statements that can not be defined as "true" and, one, referring to statements that are "not true" (negation of true). These results would suggest a type of valuation for Quantum Theory whose truth values are 1) multi-valued and 2) contextual.

Chapter 5

Why Topos?

Several approaches have been attempted to define truth values in Quantum Mechanics, one of them, which I am going to discuss, is due mainly to Professors Isham C. J. and Butterfield J [1], [9], [11], [2], [4]. The characteristic of this approach is that it makes use of a branch of mathematics called Topos Theory to define valuations for Quantum Propositions. The reason for adopting Topos Theory is the existence of an element of Topos theory called subobject classifier [14] [13] [16]. Essentially what a subobject classifier does is to define subobjects by assigning the value true to members of the subobject in question while it assigns the value false to elements not belonging to the subobject i.e. the subobject classifier classifies subobjects in terms of what elements belong to them. We have seen from the previous sections that in Quantum Mechanics (as in Classical Mechanics) propositions (or better equivalent classes of propositions) are identified with subsets of $\mathcal{H}$, such that the truth of a proposition is given in term of it belonging or not belonging to a certain subset. Specifically, a proposition X represented by projection operator $\hat{P}$ is said to be true if X belongs to the subset of $\mathcal{H}$ in which $\hat{P}$ projects, therefore, truth values can be seen as functions that assigns elements to subsets. This is precisely what a subobject classifier does in Topos theory. In order to define in details what a subobject classifier is we need to be familiar with what a Topos is. Since Topos is a particular kind of Category, in what follows I will first introduce the concept of Category and then define what a Topos is.

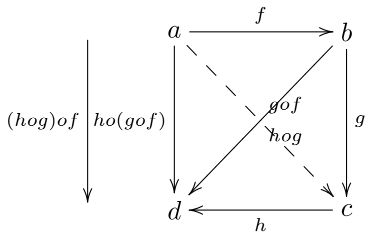
Category

Definition 5.1 *A* ***category*** *consists of two things:*

1. *a collection of objects*
2. *a collection of morphisms between these objects such that the following conditions hold:*

- ***composition condition****: given two morphisms* $f : a \to b$ *and* $g : b \to c$ *with dom g=cod f then there exists the composite map* $gof : a \to c$
- ***associative law****: given* $a \xrightarrow{f} b \xrightarrow{g} c$ *then (h o(g o f))=((h o g)o f) i.e. the following diagram commutes*

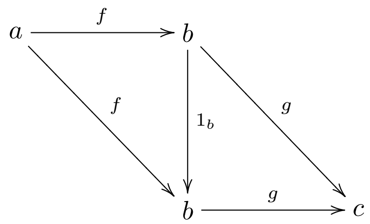
Diagram 5.1



- ***identity law****: for any object b in the category there exists a morphism* $1_b : b \to b$ *called identity arrow such that, given any other two morphisms*

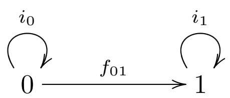
$f : a \to b$ and $g : b \to c$ we then have $1_b o f = f$ and $g o 1_b = g$, i.e. the following diagram commutes

Diagram 5.2



Simple example [13] of a two element category is the following

Example 5.1



This category has 3 arrows:

- $i_0 : 0 \to 0$ *identity on 0*
- $i_1 : 1 \to 1$ *identity on 1*
- $f_{01} : 0 \to 1$

it is easy to see that the composition arrow are: $i_0 o i_0 = i_0$,$i_1 o i_1 = i_1$,$i_1 o f_{01} = f_{01}$ *and* $f_{01} o i_1 = f_{01}$.

Following this we can define what a Topos is.

Topos

Definition 5.2 *A **Topos** is a category T with the following extra properties:*

- *T has an initial (0) and a terminal (1) object*
- *T has pullbacks*

- *T has pushouts*
- *T has exponentiation i.e. T is such that for every pair of objects X and Y in T exists the map Y^X*
- *T has a subobject classifier*

(See Definition 8.1, 8.2, 8.9, 8.10, 8.5 for the above properties) Let us now turn our attention to the Subobject Classifier.

5.1 Subobject Classifier

In order to define a what a subobject classifier is, we first need to understand what a subobject (categorical version of a subset) is, and what it means for an element to belong or not to a certain subobject.

For this purpose, let us consider a specific example in Set which is a type of Category. Given a subset A of S i.e $A \subseteq S$, the notion of being a subset can be expressed mathematically using the so called characteristic function: $\chi_A : S \to \{0, 1\}$ (Definition 8.11), which is defined as follows:

$$\chi_A(x) = \begin{cases} 0 & if \quad x \notin A \\ 1 & if \quad x \in A \end{cases} \tag{5.1}$$

(here we interpret 1=true and 0=false). The role of the characteristic function is to determine what elements belong to a certain subset.

From expression 5.1, we can deduce that there is a bijective correspondence between subsets of S, and functions $\chi_A : S \to \{0, 1\}$ from S to the set $\{1, 0\}$. In order to show that the collection of all subsets of S denoted by $\mathcal{P}(S)$, and the collection of all maps from S to the set $\{0, 1\} = 2$ denoted by 2^S are isomorphic, we need to show that the function $y : \mathcal{P}(S) \to 2^S$, which in term of single elements of $\mathcal{P}(S)$ is $A \to \chi_A$, is a bijection, i.e. we need to show that $A \to \chi_A$ is a) injective and b) surjective

Proof 5.1 *a)* y is injective (1:2:1)*: consider the case in which $\chi_A = \chi_B$ where*

$$\chi_B(x) = \begin{cases} 1 & iff \quad x \in B \\ \\ 0 & iff \quad x \notin B \end{cases}$$

It follows that since the two functions are the same, to the codomain 1 they both associate the same domain, therefore A=B

b) y is surjective (onto)*: given any function $f \in 2^S$ then there must exist a subset A of S such that $A_f = \{x : x \in D \ \ and \ \ f(x) = 1\}$ i.e. $A_f = f^{-1}(\{1\})$ therefore $f = \chi_{A_f}$*

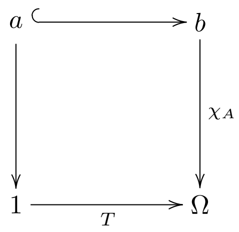
(Note, this result is valid for any category).

Now that we have defined what a subset is and proved that to each subset of a given set there corresponds a unique characteristic function which determines which elements belong to the subset, we can now define, in the general language of categories, what a subobject classifier is.

Theorem 5.1 *Given a Topos $\mathcal{T}$ with a terminal object 1 (Definition* 8.1*) then a* ***subobject classifier*** *for $\mathcal{T}$ is $\mathcal{T}$-object Ω together with a $\mathcal{T}$-arrow $true : 1 \to \Omega$ such that the following axiom is satisfied:*

Axiom 5.1 *(*Ω***-axiom****): for each monic (Definition* 8.7 *) function $f : a \to b$ there is one and only one $\mathcal{T}$-arrow χ_A, such that the following diagram is a pullback*

Diagram 5.3



It can be clearly seen from the above digram that :

$$a = {\chi_a}^{-1}(\{1\})$$

Therefore, the subobject classifier defines subobjects by assigning the value 1=*true* to the members of the subobject in question, while it assigns the value 0=*false* to elements not belonging to the subobject.

In both Classical and Quantum Logic the logical connectives can be expressed in terms of set operations between subsets of the state space inducing, in such a way, an "algebra of subobject". This entails that, given a Topos $\mathcal{T}$, each logical connective can be expressed in terms of a subobject classifier.

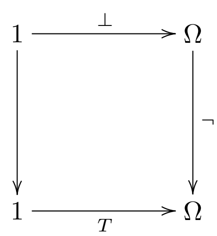
For example Goldblatt (1984) [13] illustrates negation and conjunction in classical physics as follows:

1. **Negation**: negation can be defined as the characteristic function

$$\chi_\perp = \neg : \Omega \to \Omega$$

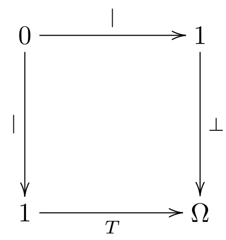
such that the following diagram is a pullback (Definition 8.9)

Diagram 5.4



Where $\perp$ is the Topos analog of the arrow *false* in Set i.e. $\perp$ is the character of $| : 0 \to 1$

Diagram 5.5



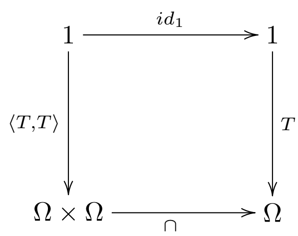
such that $\perp \cap \ T = 0_\Omega$ and (since we are considering classic la Boolean logic) $\perp \cup \ T = 1_\Omega$.

2. **Conjunction**

$$\cap : \Omega \times \Omega \to \Omega$$

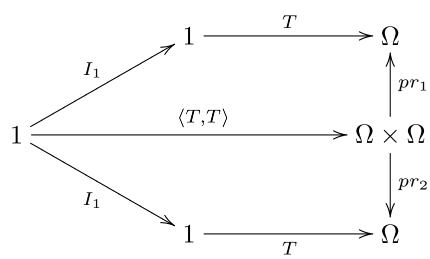
is identified with the character of the product arrow 8.8 $\langle T, T\rangle : 1 \to \Omega \times \Omega$ such that the following diagram commutes

Diagram 5.6



where $\langle T, T\rangle$ is defined as follows

Diagram 5.7



So it seem that the logical connectives can be defined without any ambiguity in terms of arrows in a given Topos.

From what stated so far, it emerges that Topos theory has the desired features to be a good candidate for developing a logical system for propositions in both Classical and Quantum mechanics.

Now the question arises naturally: what Topos do we consider such that we can assign truth values to propositions that are consistent with the mathematical formalism of Quantum Theory? As we will see in the following sections the answer to this question is : *the Topos of presheaves* [1]

5.2 Topos of Presheaves

In order to understand what a presheaf is, one needs to be familiar with functors.

Functors

Generally speaking a functor is a transformation from one category $\mathscr{C}$ to another category $\mathscr{D}$, such that the categorical structure of the domain $\mathscr{C}$ is preserved i.e. gets mapped onto $\mathscr{D}$.

There are two types of functors:

1. **Covariant Functor**

2. **Contravariant Functor**

1. **Definition 5.3** *: A* ***covariant functor*** *from a category $\mathscr{C}$ to a category $\mathscr{D}$ is a map $F : \mathscr{C} \rightarrow \mathscr{D}$ that assigns to each $\mathscr{C}$-object a $\mathscr{D}$-object $F(a)$ and to each $\mathscr{C}$-arrow $f : a \rightarrow b$ a $\mathscr{D}$-arrow $F(f) : F(a) \rightarrow F(b)$ such that the following are satisfied:*

(a) $F(1_a) = 1_{F(a)}$

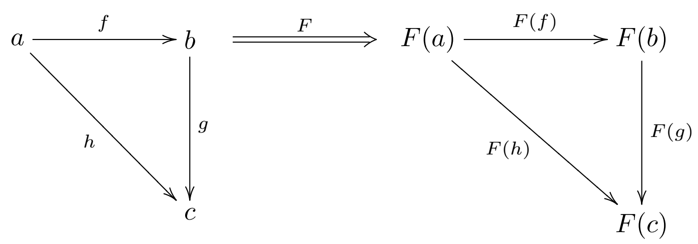
(b) $F(fog) = F(f)oF(g)$ *for any* $g : c \to a$

It is clear from the above that a covariant functor is a transformation that preserves both:

- the domain's and the codomain's identities
- the composites of functions i.e. it preserves the direction of the arrows

This can be easily seen with the aid of the following diagram;

Diagram 5.8



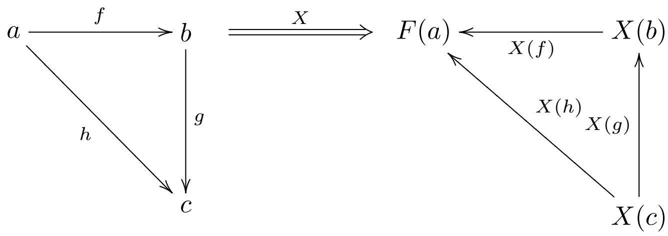
2. **Definition 5.4** *A* ***contravariant functor*** *from a category* $\mathscr{C}$ *to a category* $\mathscr{D}$ *is a map* $X : \mathscr{C} \to \mathscr{D}$ *that assigns to each* $\mathscr{C}$*-object a a* $\mathscr{D}$*-object X(a) and to each* $\mathscr{C}$*-arrow* $f : a \to b$ *a* $\mathscr{D}$*-arrow* $X(f) : X(b) \to X(a)$ *such that the following are satisfied*

   (a) $X(1_a) = 1_{X(a)}$

   (b) $X(fog) = X(g)oX(f)$ *for any* $g : c \to a$

   A diagrammatic representation of a contravariant functor is the following:

Diagram 5.9



As we can see from the above diagram a contravariant functor in mapping arrows from one category to the next it reverses the directions of the arrows by mapping domains to codomains and vice versa.

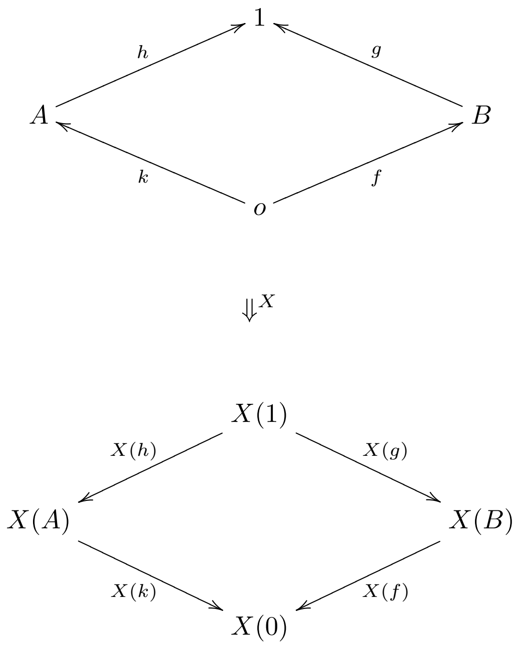
Following this we can define presheaves.

Presheaf

Definition 5.5 *A* ***presheaf*** *on a small*[1] *category* $\mathscr{C}$ *is a contravariant functor* $X : \mathscr{C} \to S$ *form* $\mathscr{C}$ *to the category of sets* S

[1] A small is a category whose collection of morphisms and objects form a genuine Set

Example 5.2 *An example of a presheaf X on a category $\mathscr{C}$ is:*

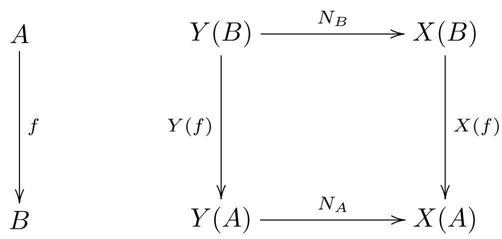


One can ascribe the status of Topos to a collection of presheaves on a category $\mathscr{C}$ by transforming the collection of presheaves into a category and, then, verifying that such a category satisfies the required conditions of being a Topos.

To transform the collection of presheaves on $\mathscr{C}$ into a category one needs to define arrows between presheaves.

Let us consider two presheaves X and Y on $\mathscr{C}$ such that $X : \mathscr{C} \rightarrow Set$ and $Y : \mathscr{C} \rightarrow Set$. The action of these presheaves is to imprint two different images of $\mathscr{C}$ on *Set*, therefore we could define an arrow between the presheaves X and Y as an arrow that compared the different images of $\mathscr{C}$ produced by the two presheaves, for example we could assign to each $\mathscr{C}$-object A, the set arrow $X(A) \rightarrow Y(A)$. A condition that should be imposed on these "image comparing arrows" between presheaves is that they should preserve the relation between the elements in $\mathscr{C}$ i.e. they should be structure preserving. Given this requirements we give the following definition of arrow between presheaves.

Definition 5.6 *A **natural transformation** from $Y : \mathscr{C} \to set$ to $X : \mathscr{C} \to set$ is an assignment of an arrow $N : Y \to X$ that associates to each object A in $\mathscr{C}$ an arrow $N_A : Y(A) \to X(A)$ in Set such that, for any $\mathscr{C}$-arrow $f : A \to B$ the following diagram commutes*



i.e.

$$N_A o Y(f) = X(f) o N_B$$

where $N_A : Y(A) \to X(A)$ are the components on N while N is the *natural transformation*. From this diagram it is clear that the two arrows N_A and N_B turn the Y-picture of $f : A \to B$ into the respective X-picture.

Now that we have turned the collection of presheaf on $\mathscr{C}$ into a category in its own right, namely $\mathcal{S}^{\mathscr{C}^{op}}$ [2] we can define the following.

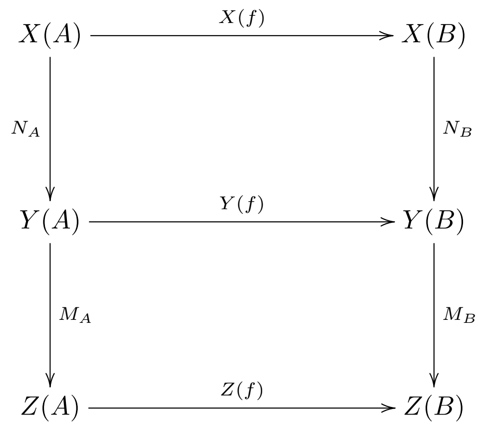
- Identity maps for objects X in $\mathcal{S}^{\mathscr{C}^{op}}$ are identified with maps i_X whose components i_{X_A} are the identity maps of X(A) in $\mathcal{S}$
- Composition maps in $\mathcal{S}^{\mathscr{C}^{op}}$: Consider X,Y and Z that belong to $\mathcal{S}^{\mathscr{C}^{op}}$ such that there, exist maps $X \xrightarrow{N} Y$ and $Y \xrightarrow{M} Z$ between them. We can then form a new map $X \xrightarrow{MoN} Z$, whose components would be

[2]note that $\mathscr{C}^{op}$ represents the opposite of the category $\mathscr{C}$. Objects in $\mathscr{C}^{op}$ are the same as the objects in $\mathscr{C}$, while the morphisms are the inverse of the morphisms in $\mathscr{C}$ i.e. $\exists$ a $\mathscr{C}^{op}$-morphisms $f : A \to B$ iff $\exists$ a $\mathscr{C}$- morphisms $f : B \to A$. This entails that we could alternatively define a presheaf on $\mathscr{C}$ as a covariant functor $X : \mathscr{C}^{op} \to Set$.

$(MoN)_A = M_A o N_A$,

i.e. graphically we would have for $f : B \to A$

Diagram 5.10



It can be proved that $\mathcal{S}^{\mathscr{C}^{op}}$ is a Topos (Definition 5.2) by defining the following.

1. Terminal and initial objects.
2. Exponential maps between any two objects of $\mathcal{S}^{\mathscr{C}^{op}}$
3. Pushouts and pullbacks.
4. Subobject classifier.

Detail of the proof is beyond the scope of this paper. Further information can be found on [13], [16], [14]. Relevant to our discussion is the terminal object and the subobject classifier since we will use them later on to define valuations in Quantum Mechanics.

Definition 5.7 *A* ***terminal object*** *in* $\mathcal{S}^{\mathscr{C}^{op}}$ *is the constant functor* $1 : \mathscr{C} \to \mathcal{S}$ *that maps every* $\mathscr{C}$*-object to the one element Set* $\{0\}$ *and every* $\mathscr{C}$*-arrow to the identity arrow on* $\{0\}$.

We shall now describe what is meant by subobject classifier in the Topos of presheaves.

5.3 Subobject Classifier in the Topos of Presheaves

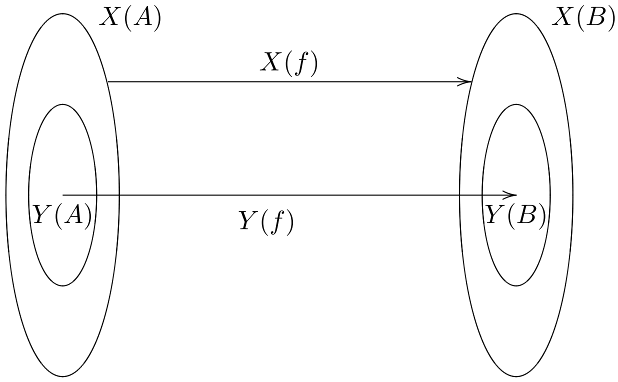
Up to now we have defined what is meant by a presheaf on a category. We then showed that it is possible to transform the collection of presheaves on a category $\mathscr{C}$ into a category in its own rights, namely $\mathcal{S}^{\mathscr{C}^{op}}$, by defining a natural transformation between presheaves. We then claimed that $\mathcal{S}^{\mathscr{C}^{op}}$ can be given the status of a Topos. This entails that there exists an element in $\mathcal{S}^{\mathscr{C}^{op}}$ which serves the role of a subobject classifier, which, as previously stated, is the essential ingredient in defining valuations in Quantum Mechanics. In order to define what a subobject classifier in the topos of presheaves is, we first need to define what a subobject of a presheaf is.

Subobject of a Presheaf

Definition 5.8 *Y is a **subobject of a presheaf** X if there exists a natural transformation $i : Y \to X$ which is defined componentwise as $i_a : Y(A) \to X(A)$ and where i_a defines a subset imbedding i.e. $Y(A) \subseteq X(A)$.*

Since Y is itself a presheaf, the maps between the objects of Y are the restrictions of the corresponding maps between the objects of X. This can be easily seen with the aid of the following diagram:

Diagram 5.11



An alternative way of expressing this condition is through the following commutative diagram:

Diagram 5.12

$$\begin{array}{ccc} Y(A) & \xrightarrow{Y(f)} & Y(B) \\ \downarrow{\scriptstyle i_A} & & \downarrow{\scriptstyle i_B} \\ X(A) & \xrightarrow[X(f)]{} & X(B) \end{array}$$

Since, as we will show later on, the subobject classifier assigns to each object belonging to the category on which it is defined the set of sieves on that object, in order to define what a subobject classifier is, we need to introduce the notion of a sieve on an object

Sieves

Definition 5.9 *A* ***sieve*** *on an object* $A \in \mathscr{C}$ *is a collection* S *of morphisms in* $\mathscr{C}$ *whose codomain is* A *and such that, if* $f : B \to A \in S$ *then, given any morphisms* $g : C \to B$ *we have* $fog \in S$.

An important property of sieves is the following: if $f : B \to A$ belongs to S which is a sieve on A, then the pullback of S by f determines a principal sieve on B, i.e.

$$f^*(S) := \{h : C \to B | foh \in S\} = \{h : C \to B\} = \downarrow B \tag{5.2}$$

The principal sieve of an object A, denoted by $\downarrow A$, is the sieve that contains the identity morphism of A therefore it is the biggest sieve on A.

We are now ready to give the definition of a subobject classifier.

Subobject Classifier

Definition 5.10 *A* ***Subobject Classifier*** Ω *is a presheaf* $\Omega : \mathscr{C} \to \mathcal{S}^{\mathscr{C}^{op}}$ *such that to each object* $A \in \mathscr{C}$ *there corresponds an object* $\Omega(A) \in \mathcal{S}^{\mathscr{C}^{op}}$ *which represents the set of all sieves on A, and to each* $\mathscr{C}$*-arrow* $f : B \to A$ *there corresponds an* $\mathcal{S}^{\mathscr{C}^{op}}$*-arrow* $\Omega(f) : \Omega(A) \to \Omega(B)$ *such that* $\Omega(f)(S) := \{h : C \to B | foh \in S\}$ *is a sieve on B, where* $\Omega(f)(S) \equiv f^*(S)$

As we saw in Theorem 5.1 Section 5.1, in order for an object of a Topos to be a subobject classifier, it has to satisfy the Ω-Axiom (Axiom 5.1) [13]. In order to prove that the definition of a Subobject Classifier in the Topos of presheaves given above, does indeed satisfy the Ω-Axiom, we need to define the analog of arrow *true* (T) and the *character function* in Topos.

Definition 5.11 $\boldsymbol{T} : 1 \to \Omega$ *is the natural transformation that has components* $T_A : \{0\} \to \Omega(A)$ *given by* $T_A(0) = \downarrow A$

To understand how T works, let us consider a monic arrow in $\mathcal{S}^{\mathscr{C}^{op}}$ $f : F \to X$ which is defined component wise $f_A : F(A) \to X(A)$ and represents subset inclusion. Now we define the character $\chi^F : X \to \Omega$ of f which is a natural transformation in the category of presheaves such that the components χ^F_A represent functions from X(A) to $\Omega(A)$, as shown in the following diagram.

Diagram 5.13

$$\begin{array}{ccc} F(A) & \xhookrightarrow{f_A} & X(A) \\ \downarrow & & \downarrow \chi_A^F \\ \{0\} & \xrightarrow{T} & \Omega(A) \end{array}$$

where $\{0\} \equiv 1$. From the above diagram we can see that χ_A^F assigns to each element x of X(A) a sieve $\Omega(A)$ on A. For a function to belong to the sieve $\Omega(A)$ on A we require that the following diagram commutes

Diagram 5.14

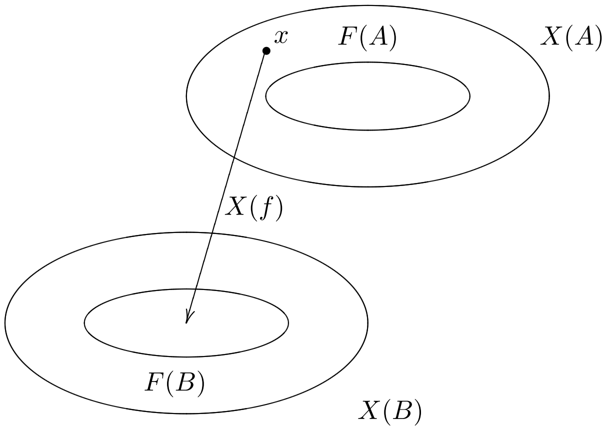
$$\begin{array}{ccc} F(A) & \hookrightarrow & X(A) \\ F(f) \downarrow & & \downarrow X(f) \\ F(B) & \hookrightarrow & X(B) \end{array}$$

Therefore

$$\chi_A^F(x) := \{f : B \to A | X(f)(x) \in F(B)\} \tag{5.3}$$

What equation 5.3 means is that we require F(f) to be the restriction of X(f) to F(A). This condition is expressed by the following diagram

Diagram 5.15



i.e. f belongs to $\Omega(A)$ iff X(f) maps x into F(B).

$\chi^F_A(x)$ as defined by equation 5.3, represents a sieve on A.

Proof 5.2 *Consider the following commuting diagram which represents subobjects F of the presheaf X.*

Diagram 5.16

$$\begin{array}{ccccc} F(A) & \xrightarrow{F(f)} & F(B) & \xrightarrow{F(g)} & F(C) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ X(A) & \xrightarrow{X(f)} & X(B) & \xrightarrow{X(g)} & X(C) \end{array}$$

If $f : B \to A$ belongs to $\chi^F_A(x)$ then given $g : C \to B$ it follows that f o g belongs to $\chi^F_A(x)$ since from diagram 5.16 it can be deduced that $X(fog)(x) \in F(C)$. This is precisely the definition of a sieve so we have proved that $\chi^F_A(x) := \{f : B \to A | X(f)(x) \in F(B)\}$ is a sieve.

Now that we have defined the analog of arrow *true* and the *character function* in Topos, we can prove that Ω is a subobject classifier. As a consequence of the Ω-Axiom the condition of being a subobject classifier can be restated in the following way.

Definition 5.12 Ω *is a* ***subobject classifier*** *iff there is a "one to one" correspondence between subobject of X and morphisms from X to* Ω

Given this alternative definition of a subobject classifier, it is easy to prove that Ω is a subobject classifier. In fact, from equation 5.3, we can see that indeed, there is a 1:2:1 correspondence between subobject of X and characteristic morphism (character) χ.

Moreover for each morphism $\chi : X \to \Omega$ we have

$$F^{\chi}(A) := \chi_A^{-1}\{1_{\Omega(A)}\}$$

$$= \{x \in X(A)|\chi_A(x) = \downarrow A\}$$

$$= \text{subobject of X}$$

The set of sieves $\Omega(A)$ for a given object A forms a Heyting algebra (see Definition 5.13 and Proof 5.4) with a null element $0_{\Omega(A)} = \emptyset$, a unit element $1_{\Omega(A)} =\downarrow A$ and a partial ordering given by subset inclusions, i.e. $S_1 \leq S_2$ iff $S_1 \subseteq S_2$, where S_1 and S_2 represents sieves on A. This entails that the logical connectives can be defined in terms of set operations:

$$S_1 \wedge S_2 := S_1 \cap S_2 = \text{greatest lower bound} \tag{5.4}$$

$$S_1 \vee S_2 := S_1 \cup S_2 = \text{least upper bound} \tag{5.5}$$

$$S_1 \Rightarrow S_2 := \{f : B \to A|\forall g : C \to B fog \in S_1 \Rightarrow fog \in S_2\} \tag{5.6}$$

$$\neg S := (S \Rightarrow 0) = \{f : B \to A|\forall g : C \to B fog \notin S\} \tag{5.7}$$

Therefore, the Heyting algebra of sieves defined for each object A forms a logic. This logic does not violate the Metalanguage/object-language distinction, i.e. the logical connectives are such that they form elements form elements of the lattice. To elaborate, a Heyting algebra is a *relative pseudo complemented lattice* (Definition 5.13) and therefore the elements *meet* (greatest lower bound), *join* (least upper bound) and the *relative pseudo complement* belong to the lattice. This entails that the composite elements, obtained by applying the logical connectives “and”, “or”, and “not” as defined by 5.4, 5.5 and 5.7 to individual elements of a Heyting algebra, will themselves be elements of the algebra. The result is not so easily deduced for the implication relation. In what follows we will prove that, given two sieves S_1 and S_2 that belong to $\Omega(A)$, $S_1 \Rightarrow S_2$, as defined by 5.6 belongs to $\Omega(A)$. Let us consider a function $f : B \to A$ such that $f : B \to A \in (S_1 \Rightarrow S_2)$ then, by definition, $\forall g : C \to B$ and $\forall h : D \to C$

$$(fo(goh)) = ((fog)oh) \in S_1 \Rightarrow (fo(goh)) = ((fog)oh) \in S_2$$

The above equation is precisely the condition for (f o g) to belong to $S_1 \Rightarrow S_2$. Since this result is valid for any g it follows that $S_1 \Rightarrow S_2$ is a sieve. The above results are very important since, as we will see in Chapter 6, the set of sieves defined on Quantum operators will represent truth values for Quantum proportions. We now define what an Heyting algebra is.

Heyting Algebra

Definition 5.13 *[13] A* ***Heyting algebra H*** *is a relative pseudo complemented distributive lattice.*

The property of being *distributive* means that the following equations are satisfied for any $S_i \in \mathbf{H}$

$$S_1 \wedge (S_2 \vee S_3) = (S_1 \wedge S_2) \vee (S_1 \wedge S_3)$$

$$S_1 \vee (S_2 \wedge S_3) = (S_1 \vee S_2) \wedge (S_1 \vee S_2)$$

The property of being a *relative pseudo complemented lattice* means that for any two elements $S_1, S_2 \in \mathbf{H}$ there exist a third element $S_3 \in \mathbf{H}$, such that:

1. $S_1 \cap S_3 \subseteq S_2$

2. $\forall S \in \mathbf{H} \quad S \subseteq S_3 \quad iff \quad S_1 \cap S \subseteq S_2$

where S_3 is defined as the *pseudo complement* of S_1 relative to S_2 i.e., the greatest element of the set $\{S : S_1 \cap S \subseteq S_2\}$, and it is denoted as $S_1 \Rightarrow S_2$.

A particular feature of the Heyting algebra is the negation operation. The negation of an element S is defined to be the pseudo-complement of S i.e. $\neg S := S \Rightarrow 0$ Therefore we can write

$$\neg S := \{f : B \to A | \forall g : C \to B, fog \notin S\}$$

The above equation entails that $\neg S$ is the least upper bound of the set $\{x : S \cap x = 0\}$, i.e. the biggest set that does not contain any element of S. From the above definition of negation operation it follows that the Heyting algebra does not satisfy the law of excluded middle [5], i.e. given any element S of an Heyting algebra we have the following relation: $S \vee \neg S \leq 1$.

Proof 5.3 *Let us consider $S \vee \neg S = S \cup \neg S$, this represents the least upper bound of S and $\neg S$ therefore, given any other element S_1 in the Heyting algebra such that $S_1 \leq S$ and $S_1 \leq \neg S$, then $S \vee \neg S \leq S_1$. But since for any S we have $S \leq 1$ and $\neg S \leq 1$ it follows that $S \vee \neg S \leq 1$.*

Now that we have defined what a Heyting algebra is we can prove what we previously stated namely, given an object A, the set of sieves $\Omega(A)$ defined on A forms a Heyting algebra. In order to prove the above, we need to show that, given any three sieves S, S_1 and S_2 that belong to $\Omega(A)$ the following condition holds: $S \subseteq (S_1 \Rightarrow S_2) \Leftrightarrow (S \cap S_1) \subseteq S_2$. The prove that follows consists of two parts the first part proves that $(S \cap S_1) \subseteq S_2$ implies $S \subseteq (S_1 \Rightarrow S_2)$ while the second part proves that $S \subseteq (S_1 \Rightarrow S_2)$ implies $(S \cap S_1) \subseteq S_2$. The first part of the proof is the following:

Proof 5.4

1. *suppose $S \cap S_1 \subseteq S_2$ then*

$$f \in (S \cap S_1) \Rightarrow ((f \in S_1) \wedge (f \in S)) \Rightarrow f \in S_2$$

From the definition of the logical connectives $\wedge$ and $\Rightarrow$ [5], it follows that the above equation is equivalent to

$$(f \in S) \Rightarrow (f \in S_1 \Rightarrow f \in S_2)$$

From the property of sieves $f \in S_1 \Rightarrow fog \in S_1 \quad \forall g$, therefore we can write

$$(fog \in S) \Rightarrow (fog \in S_1 \Rightarrow fog \in S_2) \tag{5.8}$$

However the right hand side of 5.8 means that $f \in (S_1 \Rightarrow S_2)$ (see equation 5.6*), therefore we can re-write equation 5.8 as $f \in S \Rightarrow f \in (S_1 \Rightarrow S_2)$, which implies that $S \subseteq (S_1 \Rightarrow S_2)$. We have so proved that the starting condition $(S \cap S_1) \subseteq S_2$ implies $S \subseteq (S_1 \Rightarrow S_2)$.*

The second part of the proof is the following.

2. *Suppose that $S \subseteq (S_1 \Rightarrow S_2)$ then*

$$f \in S \cap S_1 \Rightarrow (f \in S) \wedge (f \in S_1) \Rightarrow f \in (S_1 \Rightarrow S_2)$$

but, given the starting condition it follows that

$$f \in S \Rightarrow f \in (S_1 \Rightarrow S_2)$$

therefore from equation 5.6

$$fog \in S_1 \Rightarrow fog \in S_2 \qquad \forall g \tag{5.9}$$

Since equation 5.9 is valid for any g we can choose g=1 obtaining the following

$$f \in S_1 \Rightarrow f \in S_2$$

so, summing up, we have

$$f \in S \cap S_1 \Rightarrow f \in S_2$$

therefore $(S \cap S_1) \subseteq S_2$.

We have then proved that $S \subseteq (S_1 \Rightarrow S_2) \Rightarrow (S \cap S_1) \subseteq S_2$.

Putting the results of both parts of the proof together we obtain

$$S \subseteq (S_1 \Rightarrow S_2) \Leftrightarrow (S \cap S_1) \subseteq S_2$$

The fact that the set of sieves defined on an object forms a Heyting algebra is of utter most importance for the purpose of defining a valuation function in Quantum Mechanics (see Chapter 6 Section 6.2). The reasons will become clear later on in this paper, but, for the time being, we can anticipate that this feature of the set of sieves enables us to ascribe, to the set of sieves defined on particular objects, the role of truth values for propositions in Quantum Mechanics.

5.4 Global and Local sections

Other important features of Topos theory are the local and global sections.

Definition 5.14 *A **global section** or **global element** of a presheaf X in* $\mathcal{S}^{\mathscr{C}^{op}}$ *is a map* $k : 1 \to X$ *from the terminal object 1 to the presheaf X.*

What k does is to assign to each object A in $\mathscr{C}$ an element $k_A \in X(A)$ in the corresponding object of the presheaf X. The assignment is such that, given a function $B \to A$ the following relation holds

$$X(f)(k_A) = k_B \tag{5.10}$$

What 5.10 uncovers, is that the elements of X(A) assigned by the global section k, are mapped into each other by the morphisms in X. Presheaves with a local or partial section can exist even if they do not have a global section.

Definition 5.15 *A* ***local*** *or* ***partial section*** *of a presheaf X in* $\mathcal{S}^{\mathscr{C}^{op}}$ *is a map* $\rho : U \rightarrow X$ *where U is a subobject of the terminal object 1.*

In a presheaf, a subobject U of 1 can either be the empty set $\emptyset$, or a singleton $\{*\}$. From the above definition, it is clear that a local section is an assignment of an element of an object of X to the corresponding subobject U of 1 in $\mathscr{C}$. This assignment is said to be "closed downwards", i.e. given a subobject U(A)=$\{*\}$ of 1 and a $\mathscr{C}$-morphisms $f : B \rightarrow A$ then we have U(B)=$\{*\}$. To illustrate let us consider a category with 4 elements $\{A, B, C, D\}$ such that the following relations hold between the elements:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle g} \\ D & \xrightarrow[p]{} & C \end{array}$$

Given a subobject U of 1 we then have the following relations

$$\begin{array}{ccc} U(A) & \xrightarrow{U(f)} & U(B) \\ {\scriptstyle U(i)}\downarrow & & \downarrow{\scriptstyle U(g)} \\ U(D) & \xrightarrow[U(p)]{} & U(C) \end{array}$$

If U(A)=$\emptyset$ then U(f) is either the unique function $\emptyset \rightarrow \{*\}$ iff $U(B) = \{*\}$ or $\emptyset \rightarrow \emptyset$ iff $U(B) = \emptyset$. If instead $U(A) = \{*\}$ then the only possibility is that $U(B) = \{*\}$ since there does not exist a function $\{*\} \rightarrow \emptyset$. Therefore ρ assigns to particular subsets of objects $A \in X$, elements ρ_A. These objects A are called the domain of ρ ($dom\ \rho$) and are such that the following conditions are satisfied:

- the domain is closed downwards i.e. if $A \in dom\ \rho$ and if there exists a map $f : B \rightarrow A$ then $B \in dom\ \rho$

- if $A \in dom\rho$ and if there exists a map $f : B \to A$, then the following condition is satisfied:

$$X(f)(\rho_A) = \rho_B$$

5.5 Kochen-Specker Theorem defined in Topos Language

There are various ways, using Topos language, to define the Kochen-Specker Theorem. These different ways arise from the different choices of categories and presheaves used to express the theorem. In what follows we will define different types of categories that arise in Quantum Mechanics and different types of presheaves which will, then, be used to express the Kochen-Specker Theorem [1].

5.5.1 Relevant Categories

In this section we will delineate two different categories that arise in Quantum Mechanics namely the category $\mathcal{O}$ of self adjoint operators, and the category $\mathcal{W}$ of Boolean subalgebras of the lattice $P(\mathcal{H})$. We will then analyse the existing relation between them.

Category $\mathcal{O}$

Definition 5.16 *[1], [9], [11] the **Set** $\mathcal{O}$ of bounded self-adjoint operators is a **category** such that*

- *the objects of $\mathcal{O}$ are the self-adjoint operators*

- *given a function $f : \sigma(\hat{A}) \to \mathbb{R}$ (from the spectrum of $\hat{A}$ to the Reals) such that $\hat{B} = f(\hat{A})$ then there exists a morphism $f_{\mathcal{O}} : \hat{B} \to \hat{A}$ in $\mathcal{O}$ between operators $\hat{B}$ and $\hat{A}$*

To show that the category $\mathcal{O}$, so defined, is a category (see Definition 5.1), we need to show that it satisfies the identity law and composition law [13] . This can be shown in the following way:

- *Identity Law*: given any $\mathcal{O}$-object $\hat{A}$ the identity arrow is defined as the arrow $id_{\mathcal{O}_A} : \hat{A} \to \hat{A}$ that corresponds to the arrow $id : \mathbb{R} \to \mathbb{R}$ in $\mathbb{R}$.

- *Composition Condition*: given two $\mathcal{O}$-arrows $f_{\mathcal{O}} : \hat{B} \to \hat{A}$ and $g_{\mathcal{O}} : \hat{C} \to \hat{B}$ such that $\hat{B} = f(\hat{A})$ and $\hat{C} = g(\hat{B})$, then the composite function $f_{\mathcal{O}} o g_{\mathcal{O}}$ in $\mathcal{O}$ corresponds to the composite function $fog : \mathbb{R} \to \mathbb{R}$ in $\mathbb{R}$.

The category $\mathcal{O}$, as defined above, represents a pre-ordered set (Definition 8.12). In fact, the function $f : \sigma(\hat{A}) \to \mathbb{R}$ is unique up to isomorphism, therefore it follows that for any two objects in $\mathcal{O}$ there exists, at most, one morphism between them, i.e. $\mathcal{O}$ is a pre-ordered set. However, $\mathcal{O}$ fails to be a poset (Definition 8.13) since it lacks the antisymmetry property . In fact it can be the case that two operators $\hat{B}$ and $\hat{A}$ in $\mathcal{O}$ are such that $\hat{A} \neq \hat{B}$ but they are related by $\mathcal{O}$-arrows $f_{\mathcal{O}} : \hat{B} \to \hat{A}$ and $g_{\mathcal{O}} : \hat{A} \to \hat{B}$ in such a way that:

$$g_{\mathcal{O}} o f_{\mathcal{O}} = id_B \quad and \quad f_{\mathcal{O}} o g_{\mathcal{O}} = id_A \tag{5.11}$$

It is possible to transform the set of self-adjoint operators into a poset by defining a new category $[\mathcal{O}]$ in which the objects are taken to be equivalence classes of operators, whereby two operators are considered to be equivalent if the $\mathcal{O}$-morphisms relating them satisfies equation 5.11.

Category $\mathcal{W}$

Definition 5.17 *The **category** $\mathcal{W}$ of Boolean subalgebras of the lattice $P(\mathcal{H})$ has:*

- *as objects, the individual Boolean subalgebras i.e.elements $W \in \mathcal{W}$ which represent spectral algebras associated with different operators.*
- *as morphisms, the arrows between objects of $\mathcal{W}$, such that a morphism $i_{W_1 W_2} : W_1 \rightarrow W_2$ exists iff $W_1 \subseteq W_2$.*

From the definition of morphisms it follows that there is, at most, one morphisms between any two elements of $\mathcal{W}$, therefore $\mathcal{W}$ forms a poset under subalgebras inclusion $W_1 \subseteq W_2$. To show that $\mathcal{W}$ as defined above, is indeed a category, we need to define the identity arrow and the composite arrow. The identity arrow in $\mathcal{W}$ is defined as $id_W : W \rightarrow W$, which corresponds to $W \subseteq W$ whereas, given two $\mathcal{W}$-arrows $i_{W_1 W_2} : W_1 \rightarrow W_2$ $(W_1 \subseteq W_2)$ and $i_{W_2 W_3} : W_2 \rightarrow W_3$ $(W_2 \subseteq W_3)$ the composite $i_{W_2 W_3} o i_{W_1 W_2}$ corresponds to $W_1 \subseteq W_3$.

Example 5.3 *An example if the category $\mathcal{W}$ can be formed in the following way. Consider a category formed by four objects: $\hat{A}$,$\hat{B}$,$\hat{C}$,$\hat{1}$ such that the spectral decomposition is the following:*

$$\hat{A} = a_1 \hat{P}_1 + a_2 \hat{P}_2 + a_3 \hat{P}_3$$

$$\hat{B} = b_1 (\hat{P}_1 \vee \hat{P}_2) + b_2 \hat{P}_3$$

$$\hat{C} = C_1 (\hat{P}_1 \vee \hat{P}_3) + c_2 \hat{P}_2$$

then the spectral algebras are the following:

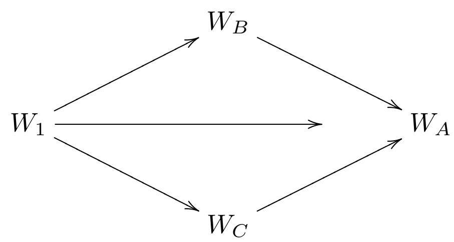
$$W_A = \{\hat{0}, \hat{P}_1, \hat{P}_2, \hat{P}_3, \hat{P}_1 \vee \hat{P}_3, \hat{P}_1 \vee \hat{P}_2, \hat{P}_3 \vee \hat{P}_2, \hat{1}\}$$

$$W_B = \{\hat{0}, \hat{P}_3, \hat{P}_1 \vee \hat{P}_2, \hat{1}\}$$

$$W_C = \{\hat{0}, \hat{P}_2, \hat{P}_1 \vee \hat{P}_3 \hat{1}\}$$

$$W_1 = \{\hat{1}\}$$

The relation between the spectral algebras is given by the following diagram:



where the arrows are subset inclusions.

Relation between categories

The categories as defined above can be related to another through the spectral algebra functor [1], [9], [11].

Definition 5.18 *The* ***spectral algebra functor*** *is a contravariant functor* W : $\mathcal{O} \to \mathcal{W}$ *such that:*

- *each object* $\hat{A} \in \mathcal{O}$ *is mapped to the object* $W_A \in \mathcal{W}$ *where* W_A *is the spectral algebra of* $\hat{A}$
- *given an* $\mathcal{O}$*-arrow* $f_{\mathcal{O}} : \hat{B} \to \hat{A}$ *then the corresponding* $\mathcal{W}$*-arrow is* $i_{W_A W_B} : W_A \to W_B$ *which is defined as subset inclusion.*

The above definition of morphisms in W as subset inclusions is motivated by the following reasoning. Let us consider an object $\hat{A} \in \mathcal{O}$, whose spectral algebra is

$W_A \in \mathcal{W}$. If there exists a map $f_{\mathcal{O}} : \hat{B} \to \hat{A}$, such that $\hat{B} = f(\hat{A})$, then from the Spectral Theorem (Theorem 3.1) it follows that the spectral algebra W_B of $\hat{B}$ is a subalgebra of W_A i.e. $W_B \subseteq W_A$. Therefore, to each map $f_{\mathcal{O}} : \hat{B} \to \hat{A}$, there corresponds a unique map $i_{W_B W_A} : W_B \to W_A$ which represents subset inclusion.

5.5.2 Relevant Presheaves and the Kochen-Specker Theorem

In this section, we describe some presheaves defined on the categories above and the different ways in which the Kochen-Specker Theorem can be expressed in terms of those presheaves. The presheaves which we will describe are the spectral presheaf on $\mathcal{O}_d$, the dual presheaf on $\mathcal{W}$ and Dual presheaf D o W.

Spectral Presheaf on $\mathcal{O}_d$

Definition 5.19 *[1]* ***spectral presheaf*** *on $\mathcal{O}_d$ (subcategory of $\mathcal{O}$ in which the operators have discrete spectra)*[3] *$\Sigma : \mathcal{O}_d \to \mathcal{S}$ is defined such that*

1. *objects $\hat{A} \in \mathcal{O}$ get mapped to $\Sigma(\hat{A}) = \sigma(\hat{A})$ where $\sigma(\hat{A})$ is the spectrum of $\hat{A}$*

2. *morphisms $f_{\mathcal{O}} : \hat{B} \to \hat{A}$ in $\mathcal{O}_d$, such that $\hat{B} = f(\hat{A})$, get mapped to $\Sigma(f_{\mathcal{O}}) : \Sigma(\hat{A}) \to \Sigma(\hat{B})$, which is equivalent to $\Sigma(f_{\mathcal{O}}) : \sigma(\hat{A}) \to \sigma(\hat{B})$ and is defined by $\Sigma(f_{\mathcal{O}})(\alpha) := f(\alpha) \quad \forall \alpha \in \sigma(\hat{A})$*

In order to prove that Σ, as defined above, is indeed a presheaf, we need to prove that, given any function $g_{\mathcal{O}} : \hat{C} \to \hat{B}$ such that $\hat{C} = g(\hat{B})$ then, the following

[3]the condition of the spectrum being discrete implies that given a function $f : \sigma(\hat{A}) \to \mathbb{R}$ then $\sigma(f(\hat{A})) = f(\sigma(\hat{A}))$

equation is satisfied:

$$\Sigma(g_{\mathcal{O}})o\Sigma(f_{\mathcal{O}}) = \Sigma(f_{\mathcal{O}}og_{\mathcal{O}})$$

Proof 5.5 *If we consider the composite function* $g_{\mathcal{O}}of_{\mathcal{O}} = h_{\mathcal{O}} : \hat{C} \to \hat{A}$ *from the definition of* Σ *we have*

$$\Sigma(f_{\mathcal{O}}) : \sigma(\hat{A}) \to \sigma(\hat{B})$$

$$\Sigma(g_{\mathcal{O}}) : \sigma(\hat{B}) \to \sigma(\hat{C})$$

$$\therefore\ \Sigma(f_{\mathcal{O}}og_{\mathcal{O}}) = \Sigma(h_{\mathcal{O}}) : \sigma(\hat{A}) \to \sigma(\hat{C})$$

therefore

$$\begin{aligned}\Sigma(f_{\mathcal{O}}og_{\mathcal{O}})(\alpha) &= \Sigma(h_{\mathcal{O}})(\alpha)\\ &= h(\alpha)\\ &= (gof)(\alpha)\\ &= go(\Sigma(f_{\mathcal{O}})(\alpha))\\ &= [\Sigma(g_{\mathcal{O}})o\Sigma(f_{\mathcal{O}})](\alpha)\end{aligned}$$

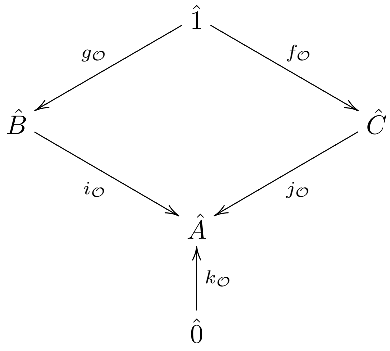
Example 5.4 *Let us Consider a simple category whose elements are defined by*

$$\hat{A} = a_1P_1 + a_2P_2 + a_3P_3$$

$$\hat{B} = b_1(P_1 \vee P_3) + b_2P_2$$

$$\hat{C} = c_1P_3 + c_2(P_1 \vee P_2)$$

This can be represented in the following diagram



The elements of the presheaf Σ *are the following:*

$$\sigma(\hat{A}) = \{a_1, a_2, a_3\}$$

$$\sigma(\hat{B}) = \{b_1, b_2\}$$

$$\sigma(\hat{C}) = \{c_1, c_2\}$$

From definition 5.19 *it follows that, for example, the map* $j_{\mathcal{O}} : \hat{C} \to \hat{A}$ *gets mapped to* $\Sigma(j_{\mathcal{O}}) : \sigma(\hat{A}) \to \sigma(\hat{C})$ *such that component wise we get the following mapping:*

$$\Sigma(j_{\mathcal{O}})(a_3) = c_1$$

$$\Sigma(j_{\mathcal{O}})(a_1) = c_2$$

$$\Sigma(j_{\mathcal{O}})(a_2) = c_2$$

In terms of the category $\mathcal{O}_d$ the Kochen -Specker Theorem can be stated as follows:

Kochen-Specker Theorem: *the spectral presheaf* $\Sigma : \mathcal{O}_d \to \mathcal{S}$ *has no global section if the Hilbert space has dimensions greater than 2.*

Motivation

A global section (Definition 5.14) of Σ is a map from the terminal object in the category of presheaves on $\mathcal{O}_d$, $\mathcal{S}^{\mathcal{O}_d^{op}}$, to Σ. Therefore, it is a map that assigns to each pair of operators $\hat{A}$ and $\hat{B}$ real numbers $k_A \in \sigma(\hat{A})$ and $k_B \in \sigma(\hat{B})$, such that, if $\hat{B} = f(\hat{A})$, then $f(k_A) = k_B$.

This definition is very similar to the definition of FUNC (as defined in equation 4.1). This suggests that one can identify the global section of Σ with a valuation $V : \mathcal{O}_d \to \mathbb{R}$ such that $V(f(\hat{A})) = f(V(\hat{A}))$.

Dual Presheaf on $\mathcal{W}$

In order to define the Kochen-Specker Theorem with respect to all operators, even those with continuous spectrum, we need to appropriately restate the above definitions in terms of the category $\mathcal{W}$.[4]

Definition 5.20 *[1] The **dual presheaf on** $\mathcal{W}$ is the presheaf $D : \mathcal{W} \to \mathcal{S}et$ such that:*

- *the objects D(W) of D are identified as the dual of the objects W of $\mathcal{W}$.i.e. $D(W) = Hom(W, \{0,1\})$, where $Hom(W, \{0,1\})$ represents the set of 2-valued homomorphism from W to $\{0,1\}$ [5].*
- *given a $\mathcal{W}$-arrow $i_{W_1W_2} : W_1 \to W_2$ the corresponding D-arrow is defined as $D(i_{W_1W_2}) : Hom(W_2, \{0,1\}) \to Hom(W_1, \{0,1\})$, which maps $\alpha \in Hom(W_2, \{0,1\})$ to its restriction $\alpha|_{W_1} \in Hom(W_1, \{0,1\})$.*

[4] recall that $\mathcal{W}$ (see definition 5.17) represents the set of all Boolean subalgebras of the lattice $P(\mathcal{H})$

[5] The reason behind the identification of objects of D with set of homomorphism from W to $\{0,1\}$ is the following: by definition, a dual of D is a map $D^* = D \to \mathbb{R}$. We know from definition of spectral projectors that $\hat{P} = \hat{P^2}$ for all $P \in W$. This implies that the only values that $\hat{P}$ can have are 1 or 0 therefore $D^* = D \to \{0,1\}$

With respect to the presheaf D the Kochen-Specker Theorem can be stated as follows.

Kochen-Specker Theorem: *the dual presheaf* $D : \mathcal{W} \to \mathcal{S}et$ *has no global section if* $dim(\mathcal{H}) > 2$.

Motivation

With respect to the presheaf D, a global section is an assignment to each element $W \in \mathcal{W}$, of an element $\alpha \in Hom(W, \{0,1\})$, such that, if $W_1 \subseteq W_2$ with $i_{W_1 W_2} : W_1 \to W_2$, then, a global section assigns to W_1 the element $D[(i_{W_1 W_2})](\alpha) = \alpha|_{W_1}$, where $\alpha|_{W_1}$ denotes the restriction of α to W_1. Therefore, for all $\hat{P}_i \in W_1$, $D[(i_{W_1 W_2})](\alpha)(\hat{P}_i) = \alpha|_{W_1}(\hat{P}_i)$. This means that a global section would assign the value of 1 to one of the atoms in a Boolean lattice $P(\mathcal{H})$ of projector operators while, the other atoms get assigned the value 0. Since all other projectors that, with respect to the lattice ordering, cover [6] the atoms are of the form $P_1 \vee P_2$, with P_1 and P_2 being the atoms, it follows that all projectors covering the atom whose value is assigned 1 also get assigned the value 1, while all the rest gets assigned the value 0. So one can associate a global section to a 2-valued homomorphism h:$P(\mathcal{H}_n) \to P(\mathcal{H}_1)$. As we recall from section 3.1.2 this type of homomorphism is prohibited by the Kochen-Specker Theorem, therefore it follows that one can associate a valuation for Quantum propositions with a global section of the dual presheaf D.

Dual Presheaf D o W

Another way of defining the Kochen-Specker Theorem with respect to all operators is by stating the theorem in terms of the global section of the Dual presheaf

[6]an element x of a lattice X is said to cover another element y iff $y < x$ and there is no other element z of X such that $y < z < x$ [4]

D o W.

Definition 5.21 *the* ***Dual presheaf*** $DoW : O \to \mathcal{S}et$ *on* $\mathcal{O}$ *is defined as a contravariant functor such that*

- *the objects* $DoW(\hat{A})$ *correspond to the dual of* W_A*, i.e.* $DoW(\hat{A}) = set(Hom(W_A, \{1,0\}))$ *the set of all homomorphism from the spectral algebra* W_A *to the Boolean algebra* $\{1,0\}$*.*

- *given* $f_{\mathcal{O}} : \hat{B} \to \hat{A}$ *such that* $\hat{B} = f(\hat{A})$ *then the corresponding morphism in D o W is* $DoW(f_{\mathcal{O}}) : D(W_A) \to D(W_B)$ *where* $DoW(f_{\mathcal{O}})(\chi) := \chi|_{W_{f(A)}}$ *i.e.* $DoW(f_{\mathcal{O}})(\chi)$ *restricts the elements of* $D(W_A)$ *to the subalgebra* $W_{f(A)} \subseteq W_A$

In terms of the dual presheaf D o W, the Kochen-Specker Theorem corresponds to the following.

Kochen-Specker Theorem: *If the Hilbert space* $\mathcal{H}$ *has dimensions greater than 2 then the dual presheaf D o W has no global section.*

Motivation

A global section γ of D o W is an assignment of an element $\gamma_A \in Hom(W_A, \{0,1\})$ to each element $\hat{A} \in \mathcal{O}$, such that, given a $\mathcal{O}$-map $f_{\mathcal{O}} : \hat{B} \to \hat{A}$, then $\gamma_B = \gamma_A|_{W_B}$, i.e. γ_B is the restriction of γ_A to W_B. Therefore the following equality is satisfied

$$DoW(f_{\mathcal{O}})(\gamma_A) = \gamma_B \tag{5.12}$$

but 5.12 is very similar to the FUNC condition namely

$$V(f(\hat{A})) = f(V(\hat{A}))$$

for any $\hat{B} = f(\hat{A})$. Therefore, there is a clear association between the global section of D o W and the valuation function.

Chapter 6

The need of “a non-Boolean logic”

The formulations of the Kochen-Specker Theorem given in Section 5.5.2, although different, lead to the same result, namely, the impossibility of using 2 valued Boolean Logic as the target logic in evaluating Quantum Propositions.
In fact, it would seem that if one adopts a Boolean Logic as the logic of the space of truth values, then, it is impossible to assign simultaneously truth values to all operators, satisfying, at the same time, the FUNC condition. The the need to adopt a multi-valued logic in Quantum Theory emerges. Moreover, this claim is ulteriorly justified by the fact that, Quantum Mechanics exhibits two types of “false” values (see Chapter 3): one referring to propositions that can not be defined as “true”, and one referring to propositions that are “not true”.
A possible candidate for a multi-valued logic in Quantum Theory would be the sieved valued logic of presheaves (see Section 5.3), in which the target logic is the

Heyting algebra of sieves defined for each element of the chosen presheaf. Each such element is called a stage of truth. This logic has as truth values the set of sieves defined at each stage of truth therefore, it satisfies the requirements of being multi-valued and contextual (see Section 4.2). Adopting the sieved valued logic of Presheaves as the logic in Quantum Mechanics, valuations in Quantum Theory become maps from Boolean algebra to Heyting algebra, i.e. truth values assigned to Quantum Propositions belong to a Heyting algebra, while the Quantum Propositions are related by a Boolean algebra. The logical structure of the domain and the logical structure of the codomain (target) of any valuation, must be related in a specific way. This leads to the following definition [1], [9].

Definition 6.1 : *A* ***valuation*** *of a Boolean algebra B in Heyting algebra H is a map $\theta : B \to H$ which satisfies the following conditions:*

- ***Null Proposition Condition****:*

$$\theta(0_B) = 0_H$$

- ***Monotonicity****:*

$$\alpha \leq_B \beta \Rightarrow \theta(\alpha) \leq_H \theta(\beta)$$

- ***Exclusivity****:*

$$\alpha \wedge \beta = 0_B, \theta(\alpha) = 1_H \Rightarrow \theta(\beta) < 1$$

- ***Unity Proposition Condition****:*

$$\theta(1_B) = 1_H$$

From the above conditions it follows that $\theta : B \to H$ satisfies two other conditions:

1. *Disjunctive Condition*

$$\theta(\alpha) \vee \theta(\beta) \leq \theta(\alpha \vee \beta)$$

Proof 6.1 *from the monotonicity condition it follows that*

$$\alpha \leq_B (\alpha \vee \beta) \Rightarrow \theta(\alpha) \leq_H \theta(\alpha \vee \beta)$$

$$\beta \leq_B (\alpha \vee \beta) \Rightarrow \theta(\beta) \leq_H \theta(\alpha \vee \beta)$$

$$\therefore$$

$$\theta(\alpha) \vee \theta(\beta) \leq_H \theta(\alpha \vee \beta)$$

2. *Conjunctive Condition*

$$\theta(\alpha \wedge \beta) \leq \theta(\alpha) \wedge \theta(\beta)$$

Proof 6.2 *from the monotonicity condition it follows that*

$$(\alpha \wedge \beta) \leq_B \alpha \Rightarrow \theta(\alpha \wedge \beta) \leq_H \theta(\alpha)$$

$$(\alpha \wedge \beta) \leq_B \beta \Rightarrow \theta(\alpha \vee \beta) \leq_H \theta(\beta)$$

$$\therefore$$

$$\theta(\alpha \wedge \beta) \leq_H \theta(\alpha) \wedge \theta(\beta)$$

In Section 5.5.2 we showed that it was possible to define valuations in terms of global sections of presheaf D o W, D and Σ. This valuations though, fell into the Kochen-Specker "no go" theorem, since the elements of these global sections could have as values only "true"=1 or "false"=0. In this section we showed that the sieved-valued logic of presheaves is a good candidate for a logic in Quantum Mechanics since it is multi-valued and contextual. Therefore the question is: which presheaf can be used to define valuations for Quantum Propositions such that they have as truth values sieves defined on the appropriate objects? The answer, surprisingly enough, stems directly from the Kochen-Specker Theorem itself. In fact, even if the Kochen-Specker Theorem prohibits truth assignment to propositions of the form $A \in \Delta$, such that they satisfy FUNC, it nevertheless allows the possibility of assigning truth values to propositions which are generalisations of $A \in \Delta$, assigning, in this respect, a partial truth to $A \in \Delta$. These generalised propositions are called coarse granings [1], [9]. Specifically, if we consider a proposition "$A \in \Delta$" and a Borel function $f : \sigma(\hat{A}) \rightarrow \mathbb{R}$, then it is possible that f is such that the proposition "$f(A) \in f(\Delta)$" can be evaluated as being true, even if "$A \in \Delta$" cannot. This is possible because "$f(A) \in f(\Delta)$" represents a generalisation of "$A \in \Delta$" (Section 3.1.1). This would suggest that we could give a truth value to propositions of the form "$A \in \Delta$" in terms of their coarse grainings "$f(A) \in f(\Delta)$". Therefore, the appropriate presheaf to define valuations in Quantum Mechanics is the coarse graining presheaf.

6.1 Coarse Graining Presheaf

From the above section we deduce that valuations of propositions of the form "$A \in \Delta$" can be given in terms of their coarse granings "$f(A) \in f(\Delta)$". But, the projector $P_{A\in\Delta}$ representing proposition "$A \in \Delta$" belongs to a different spectral algebra as does the projector $P_{f(A)\in f(\Delta)}$ which represents the proposition "$f(A) \in f(\Delta)$". Therefore, in order to assign valuations to propositions in terms of their coarse grainings, one requires to compare propositions which are related to different spectral algebras. This is precisely what the coarse graining presheaf does. In fact the coarse graining presheaf defines a logical ordering between proposition, where the logical ordering is given in terms of the scope of the implications of a proposition. For example: the proposition $A \in \Delta$ implies its coarse graning $f(A) \in f(\Delta)$ since, intuitively, if $A \in \Delta$ is true, then $f(A) \in f(\Delta)$ is also necessarily true. Therefore, $f(A) \in f(\Delta)$ is considered weaker than $A \in \Delta$ since the former is contained within the scope of implications of the latter (Section 3.1.1). It follows that the coarse granings of a proposition X belong to the group of implications of X. To clarify, let us consider a proposition of the form "$A \in \Delta$" whose associated projection operator is $\hat{E}[A \in \Delta] = \hat{P}_{A\in\Delta}$. Given a Borel function $f : \sigma(\hat{A}) \rightarrow \mathbb{R}$ such that $\hat{B} = f(\hat{A})$ then, from the Spectral Theorem (Theorem 3.1), it follows that

$$\hat{E}[A \in \Delta] \leq \hat{E}[f(A) \in f(\Delta)] \equiv \hat{P}_{A\in\Delta} \leq \hat{P}_{f(A)\in f(\Delta)} \tag{6.1}$$

However, since $\hat{P}_{A\in\Delta}$ belongs to spectral algebra W_A, whereas $\hat{P}_{f(A)\in f(\Delta)}$ belongs to spectral algebra $W_{f(A)}$, then equation 6.1 implies that we "projected" the operator $\hat{P}_{f(A)\in f(\Delta)}$ from its spectral algebra $W_{f(A)}$ to W_A, so that the partial order (<) in equation 6.1 is defined with respect to W_A. This is mathematically translated as

$$\hat{P}_{A\in\Delta} \leq i_{W_{f(A)}W_A}\hat{P}_{f(A)\in f(\Delta)}$$

The map $i_{W_{f(A)}W_A} : W_{f(A)} \rightarrow W_A$ in the above equation represents an inclusion map i.e. $W_{f(A)} \subseteq W_A$. Therefore " implication ordering" of propositions, such as $A \in \Delta$, is given in terms of the partial ordering of their respective spectral algebras, where partial ordering is identified as subset inclusion. We can then reiterate that the conceptual motivation behind the definition of the coarse graning presheaf G, is the need to be able to compare propositions which are related to different spectral algebras, so as to define " implication ordering" between them.

In what follows we will describe two different types of coarse graning presheaf G and Θ which are defined respectively on the category $\mathcal{O}$, and on the category $\mathcal{W}$ [1], [9], [11].

Coarse graining presheaf G

Definition 6.2 *A* ***coarse graining presheaf*** *on $\mathcal{O}$ is the contravariant functor $G : \mathcal{O} \rightarrow Set$ such that*

- *each object $\hat{A} \in \mathcal{O}$ gets mapped to its spectral algebra W_A i.e. $G(\hat{A}) = W_A$*
- *given an $\mathcal{O}$-map $f_{\mathcal{O}} : \hat{B} \rightarrow \hat{A}$ the corresponding function in G is $G(f_{\mathcal{O}}) : W_A \rightarrow W_B$ such that*

$$G(f_{\mathcal{O}})(\hat{E}[A \in \Delta]) = \hat{E}[f(A) \in f(\Delta)]$$

or alternatively

$$G(f_{\mathcal{O}})(\hat{P}_{A\in\Delta}) = \hat{P}_{f(A)\in f(\Delta)}$$

where $\hat{P}_{A\in\Delta} \in W_A$ and $\hat{P}_{f(A)\in f(\Delta)} \in W_B$

In order to show that G, as defined above, is indeed a presheaf (Definition 8.12), we need to show that

$$G(f_{\mathcal{O}} o g_{\mathcal{O}}) = G(g_{\mathcal{O}}) o G(f_{\mathcal{O}})$$

for $f_{\mathcal{O}} : \hat{B} \rightarrow \hat{A}$ and $g_{\mathcal{O}} : \hat{C} \rightarrow \hat{B}$

Proof 6.3

$$G(f_{\mathcal{O}} o g_{\mathcal{O}})(\hat{P}_{A\in\Delta}) = \hat{P}_{f(g(A))\in f(g(\Delta))} =$$

$$\hat{P}_{(fog)(A))\in(fog)(\Delta))} = G(g_{\mathcal{O}})(\hat{P}_{f(A)\in f(\Delta))} =$$

$$G(g_{\mathcal{O}})[G(f_{\mathcal{O}}(\hat{P}_{A\in\Delta})] = [G(g_{\mathcal{O}}) o G(f_{\mathcal{O}})](\hat{P}_{A\in\Delta})$$

Since this result is independent of $\hat{P}_{A\in\Delta}$ then $G(f_{\mathcal{O}} o g_{\mathcal{O}}) = G(f_{\mathcal{O}}) o G(g_{\mathcal{O}})$ follows. From the above definition it is clear that the map $G(f_{\mathcal{O}})$ is an inclusion map. In fact, given any two spectral algebras W_A and W_B such that $W_B \subseteq W_A$ ($\hat{B} = f(\hat{A})$), then $G(f_{\mathcal{O}}) : W_A \to W_B$ represent the embedding of the spectral algebra $W_{f(A)} = W_B$ into the spectral algebra W_A. In terms of the lattice of operators, the action of $G(f_{\mathcal{O}})$ is to move from one operator to the one immediately above it, i.e. to the operator that covers it, where the latter operator represents a weaker statement than the former operator.

Coarse Graining Presheaf Θ

One can alternatively define a coarse graining presheaf directly on the category $\mathcal{W}$.

Definition 6.3 *A* ***coarse graining presheaf*** *on $\mathcal{W}$ is a contravariant functor $\Theta : \mathcal{W} \to Set$ such that:*

- *each object $W \in \mathcal{W}$ gets mapped to itself i.e. $\Theta(W) := W$*
- *given a $\mathcal{W}$-morphism $i_{W_B W_A} : W_B \to W_A$ such that $W_B \subseteq W_A$ the corresponding Θ-morphism is $\Theta(i_{W_B W_A}) := \theta_{W_A W_B} : W_A \to W_B$*

The prove that Θ, as defined above, is indeed a presheaf, is very similar to the prove given for G therefore, for the sack of brevity, I will not report it.

6.2 Valuations

As stated at the beginning of this Chapter, a valuation which had as truth values sets of sieves would enable us to "overcome" the Kochen-Specker "no go" Theorem. Hereafter we are going to define two types of such valuations :

- **Partial Valuation**
- **Global Valuation**

The first type of valuation is related to local sections while the second type is related to global sections of the appropriate presheaf.

Although the Kochen-Specker Theorem denies the possibility of assigning global sections to certain presheaves in Quantum Theory, it nevertheless allows local section to be defined on those same presheaves, since, as we recall from Definition 5.15, local sections are defined with respect to a particular domain. To elaborate, the Kochen-Specker Theorem allows assignment of truth values on all operators belonging to the subset of commuting operators (partivular domain) of $\mathcal{O}$, such that FUNC is satisfied. Therefore, it is possible to define a valuation function which has a restricted applicability domain. This valuation is called partial valuation.

A global valuation instead, as the name suggests, is defined on all operators in $\mathcal{O}$.

6.2.1 Partial Valuations

Definition 6.4 *A* ***partial valuation*** *is a map* $V : domV \rightarrow \mathbb{R}$ *with* $V \subseteq \mathcal{O}_d$ [1] *such that the following conditions are satisfied:*

1. $\hat{A} \in domV \quad \Rightarrow \quad V(\hat{A}) \in \sigma(\hat{A})$
2. *if* $\forall \hat{A} \in domV \quad \hat{B} = f(\hat{A})$ *then a)* $\hat{B} \in domV$ *and b)* $V(\hat{B}) = f(V(\hat{A}))$.

where condition (b) is FUNC.

From the above definition, it follows that you can define a partial valuation

[1] where $\mathcal{O}_d$ is the set of bounded self-adjoint operators with discreate spectrum

function on $\mathcal{H}$, such that it respects FUNC within its domain of applicability.

How does this definition of partial valuation relate to the notion of local section? Let us consider the presheaf Σ (Definition 5.19). With respect to Σ, a local valuation is defined as a partial section of Σ, i.e. a morphism $y : U \to \Sigma$ from a subset of the terminal object $1 : \mathcal{O}_d \to \mathcal{S}et$ (Definition 8.1) to an element of Σ. As we have seen in the definition of a local section (Definition 5.15), these morphisms y are defined by a particular domain ($dom\ \rho$). Therefore a local section of Σ assigns to a certain subset of objects $\hat{A}$ in $\mathcal{O}_d$ (domain of applicability of y) an element $\sigma(\hat{A})$ of Σ, such that the domain of y is "closed downwards" [2] and $\Sigma(f)y_{\hat{A}} = y_{\hat{B}}$ holds. It follows, that the partial valuation $V : domV \to \mathbb{R}$, where domV is the set of self-adjoint discrete operators by which the subset U of 1 is defined, is equivalent to the local section $y : U \to \Sigma$. Particularly one would have $domV \to U \to \Sigma$, and, therefore the map V is such that $V(\hat{A}) \in \sigma(\hat{A})$, i.e.V assigns to each self-adjoint operator $\hat{A} \in domV$ a value that lies within its spectrum.

Since propositions are defined with respect to spectral projectors, one needs to define the action of V on spectral projectors. Let us consider the following diagram

$$\begin{array}{ccc} \Delta & \overset{f}{\hookrightarrow} & \sigma(\hat{A}) \\ \downarrow & & \downarrow \chi_{\Delta} \\ 1 & \underset{T}{\longrightarrow} & \Omega \end{array}$$

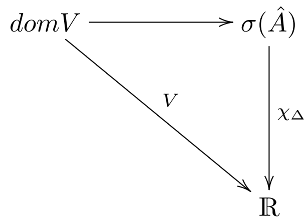
i.e. χ_{Δ} is the characteristic function of f. Since the projector $\hat{E}[A \in \Delta]$ projects onto the eigenspace associated with the Borel subset Δ of the spectrum $\sigma(\hat{A})$

[2]Domain of y $dom\ \rho$ is said to be closed downwards if $A \in dom\ \rho$ and $f : B \to A$, then $B \in dom\ \rho$

of A, it follows that $\hat{E}[A \in \Delta] = \chi_{\Delta}(\hat{A})$ (see equation 4.7). As a consequence, condition (b) in the definition of a partial valuation can be expressed as:

$$V(\hat{E}[A \in \Delta]) = \chi_A(V(\hat{A})) = \begin{cases} 1 & if \quad V(\hat{A}) \in \Delta \\ \\ 0 & otherwise \end{cases}$$

which corresponds to the following diagram



It is possible to extend the definition of partial valuation so to include all operators, even those with continuous spectrum. In this endeavor, care is needed, because of the difficulties in giving meaning to statements of the form A=a, which describe precise values for observables, when the spectrum of observables is continuous. In fact, in this case, all that can be derived from the Quantum Mechanical formalism are probabilities for infinitesimal ranges of values for observables.

It is worth noting at this point that, from Definition 6.4, real multiples of the unit operator $\hat{1}$ belong to dom V, provided $domV \neq= \emptyset$. Explicitly let us consider a constant map $c_r : \sigma(\hat{A}) \to \mathbb{R}$ such that $c_r(a) := r \quad \forall a \in \sigma(\hat{A})$. If $\hat{A} \in domV$, then from condition 2 in Definition 6.4 it follows that $c_r(\hat{A}) \in domV$. However $r\hat{1} = c_r(\hat{A})$, therefore $r(\hat{1}) \in domV$. This entails that $V(r(\hat{1})) = r(V(\hat{1})) = V(c_r(\hat{A})) = c_r(V(\hat{A})) = c_r(a) = r$, which implies that $V(\hat{1}) = 1$. What this result uncovers is that a partial valuation as defined in Definition 6.4 is never empty, since real multiples of the unit operator will always belong to dom V, therefore propositions "$A \in \Delta$" can never be said to be totally false. A way round this is to replace the category $\mathcal{O}_d$ with the category $\mathcal{O}_d^*$ which has the same characteristics as $\mathcal{O}_d$ but it doesn't admit objects of the type $r\hat{1}$ or morphisms which have these objects as domains.

Let us now consider an example of partial valuation [1].

Example 6.1 *Given a self-adjoint operator $\hat{M}$ with purely discrete spectrum and an eigenvalue $m \in \sigma(\hat{M})$, then the partial valuation $V^{M,m}$ associated with $\hat{M}$ is such that the following conditions are satisfied:*

1.

$$domV^{M,m} :=\downarrow \hat{M} := \{f_{\mathcal{O}_d} : \hat{A} \rightarrow \hat{M}\} = \{\hat{A} | \exists \;\; f \;\; \text{such that} \;\; \hat{A} = f(\hat{M})\}$$

2.

$$\hat{A} \in domV^{M,m}, \hat{A} = f(\hat{M}) \Rightarrow V^{M,m}(\hat{A}) := f(m)$$

$V^{M,m}$ so defined satisfies conditions for partial valuation.

Proof 6.4 *If $\hat{A} \in domV^{M,m}$ then, from condition 1 above it follows that $\hat{A} = f(\hat{M})$ while, from condition 2 we have $V^{M,m}(\hat{A}) := f(m)) \in \sigma(\hat{A})$. These equalities satisfies the first requirement for partial valuation.*

Moreover from 1 and 2 above, it follows that if $\hat{A} \in domV^{M,m}$ and $\hat{B} = g(\hat{A})$, then $\hat{B} = (gof)(\hat{M})$. Therefore $\hat{B} \in domV$ and $V^{M,m}(\hat{B}) = g(f(m)) = g(V^{M,m})(\hat{A})$, which means that also condition 2 for partial valuations is satisfied.

If we had statements of the form $A \in \Delta$ the action of $V^{M,m}$ would be the following:

$$V^{M,m}(A \in \Delta) := \begin{cases} \text{“}true\text{”} = 1 & \text{if } \; V^{M,m}(\hat{A}) \in \Delta \\ \\ \text{“}false\text{”} = 0 & \text{otherwise} \end{cases}$$

i.e. the proposition "$A \in \Delta$" is said to be true iff $A \in domV^{M,m}$ and $V^{M,m}(\hat{A}) \in \Delta$, while it is said to be false iff $A \in domV^{M,m}$ but $V^{M,m}(\hat{A}) \notin \Delta$. If $A \notin domV^{M,m}$ then the proposition "$A \in \Delta$" is neither true nor false.

6.2.2 Generalized Valuations

Keeping in mind the various presheaves and categories in Quantum Theory previously defined, we will now define 3 versions of a generalized valuation each of which is defined on a different category.

valuation on $\mathcal{O}$

Definition 6.5 *A **generalised valuation** on Quanutm propositions of the form $A \in \Delta$ (where Δ is a Borel subset of $\sigma(\hat{A})$) is a map $v : \mathcal{O} \rightarrow \Omega$ such that to each element $\hat{A} \in \mathcal{O}$ it assigns a sieve $v(A \in \Delta) \in \Omega(\hat{A})$ on $\hat{A}$.*

v is such that the following conditions are satisfied.

- **Functional Composition Condition (FUNC).**

  Given any Borel function $h : \sigma(\hat{A}) \rightarrow \mathbb{R}$ we have

  $$v(h(A) \in h(\Delta)) = h^*_{\mathcal{O}}(v(A \in \Delta))$$

  where $h_{\mathcal{O}} := h(\hat{A}) \rightarrow \hat{A}$ and $h^*_{\mathcal{O}}$ is the pullback of the sieve $v(A \in \Delta)$ on $\hat{A}$ to the sieve on $h(\hat{A})$.

- **Null Proposition Condition**

  $$v(A \in \emptyset) = 0_A$$

  with 0_A=empty sieve on $\hat{A}$

- **Monotonicity**

$$\Delta_1 \subseteq \Delta_2 \Rightarrow v(A \in \Delta) \leq v(A \in \Delta_2)$$

- **Exclusivity**

$$\Delta_1 \cap \Delta_2 = \emptyset \quad v(A \in \Delta_1) = true_A \quad \Rightarrow \quad v(A \in \Delta_2) \leq true_B$$

where $true_A = 1_{\Omega(\hat{A})} =\downarrow \hat{A}$=principal sieve on $\hat{A}$

- **Unit Proposition Condition**

$$v(A \in \sigma(\hat{A})) = true_A$$

From the definition of the *functional composition condition* (FUNC) given above, it appears that the truth value of propositions of the form "$A \in \Delta$" is determined in terms of the coarse graining proposition "$h(A) \in h(\Delta)$" (for a given function $h_{\mathcal{O}} := h(\hat{A}) \to \hat{A}$), such that they are evaluated as being totally true at their own " stage of truth". Explicitly, the pullback of a sieve by a function, that is also a member of that sieve, determines a *principal sieve*, i.e. the biggest sieve on an element (specifically the sieve that contains all functions whose codomain is the element on which the sieve is defined). Therefore, given a function $h : \sigma(\hat{A}) \to \mathbb{R}$, such that $h(A) \in h(\Delta)$ is a coarse graining of the proposition $A \in \Delta$, then if h belongs to the sieve, the definition of FUNC implies that $v_{h(A)}(\hat{E}[h(A) \in h(\Delta)] =\downarrow h(\hat{A})$, i.e. the coarse graining proposition $h(A) \in h(\Delta)$ is totally true at the stage of truth $h(\hat{A})$.

Note that, when dealing with the category $\mathcal{O}$ of all self-adjoint operators, one encounters situations where, given an operator with a continuous spectrum, $f(\Delta)$ is not a Borel subset of $\sigma(f(\hat{A}))$. In these cases one needs to redefine what is meant by the projector $\hat{E}[f(A) \in f(\Delta)] \equiv \hat{P}_{f(A)\in f(\Delta)}$. Namely we have

$$\hat{E}[f(A) \in f(\Delta)] := inf_{K\in\sigma(f(\hat{A}))}\{\hat{E}[f(A) \in K]|\hat{E}[A \in \Delta] \leq \hat{E}[f(A) \in K]\} \tag{6.2}$$

which is equivalent to

$$\hat{P}_{f(A)\in f(\Delta)} := inf_{K\in\sigma(f(\hat{A}))}\{\hat{P}_{f(A)\in K}|\hat{P}_{A\in\Delta} \leq \hat{P}_{f(A)\in K}\} \tag{6.3}$$

K is a Borel subset of $\sigma(f(\hat{A}))$, therefore equation 6.3 means that $\hat{P}_{f(A)\in f(\Delta)}$ is identified with the smallest projector $\hat{P}_{f(A)\in K}$ in $W_{f(A)}$ such that $\hat{P}_{A\in\Delta} \leq \hat{P}_{f(A)\in f(\Delta)}$ i.e. the projector $\hat{P}_{f(A)\in f(\Delta)}$ represents a statement $(f(A)\inf(\Delta))$ which is weaker than the statement represented by $\hat{P}_{A\in\Delta}$ "$A \in \Delta$".

Valuation on $P(\mathcal{H})$

We could alternatively define a valuation on the lattice of projections $P(\mathcal{H})$

Definition 6.6 *A **generalized valuation** on the lattice of projections $P(\mathcal{H})$ is a collection of maps $v_A := W_A \to \Omega(\hat{A})$, one for each $\hat{A} \in \mathcal{O}$, such that the following conditions are satisfied.*

- ***Functional Composition***

  Given any Borel function $h : \sigma(\hat{A}) \to \mathbb{R}$ we have

  $$v_{h(A)}(\hat{E}[h(A) \in h(\Delta)]) \equiv v_{h(A)}(\hat{P}_{h(A)\in h(\Delta)}) = h^*_{\mathcal{O}}(v_A(\hat{E}[A \in \Delta]) \equiv h^*_{\mathcal{O}}(\hat{P}_{A\in\Delta})$$

- ***Null Proposition Condition***

  $$v_A(\hat{0}) = 0_A$$

- ***Monotonicity***

  $$\hat{\alpha}, \hat{\beta} \in W_A \quad with \quad \hat{\alpha} \leq \hat{\beta} \Rightarrow v_A(\hat{\alpha}) \leq v_B(\hat{\beta})$$

- ***Exclusivity***

$$\hat{\alpha}, \hat{\beta} \in W_A \quad with \quad \hat{\alpha}\hat{\beta} = 0, v_A(\hat{\alpha}) = true_A \Rightarrow v_A(\hat{\beta}) < true_A$$

where $\hat{\alpha}\hat{\beta} = 0$ *means that the projectors* $\hat{\alpha}$ *and* $\hat{\beta}$ *are orthogonal to each other.*

- ***Unit Proposition Condition***

$$v_A(\hat{1}) = true_A$$

Valuation on $\mathcal{W}$

If, instead, we considered the category $\mathcal{W}$ then the definition of generalised valuation would be the following.

Definition 6.7 *Given the category* $\mathcal{W}$ *and the coarse graining presheaf* Θ *defined on it, the* ***generalized valuation*** *associated with* $\mathcal{W}$ *is defined as a family of local valuations* $\phi_W : W \to \Omega(W)$ *such that the following condition is satisfied:*

$$W_2 \subseteq W_1 \qquad \Rightarrow \qquad \phi_{W_2}(\theta_{W_1 W_2}(\hat{\alpha})) = i^*_{W_1 W_2}(\phi_{W_1}(\hat{\alpha})) \tag{6.4}$$

$\forall \alpha \in W_1$

Similarly to v, the truth value of a projector $\alpha \in W_1$ at stage of truth W_1, is given in terms of the coarse grained projectors $\theta_{W_1 W_2}(\hat{\alpha}) \in W_2$, which according to equation 6.4 are evaluated as being totally true at stage of truth W_2.

Subobject Classifier Ω and Generalised Valuations

The aim of this section is to define a relation between v and Ω.

In order to define such a relation we need to recall the action of the course graning presheaf G defined in Section 6.1, namely:

$$G(f_{\mathcal{O}})(\hat{P}_{A\in\Delta}) := \hat{P}_{f(A)\in f(\Delta)} \tag{6.5}$$

Now let us consider the functional composition principle. A consequence of FUNC is that, given any function $f_{\mathcal{O}} : \hat{B} \to \hat{A}$, and a valuation v, we have the following relation:

$$v_B(\hat{P}_{f(A)\in f(\Delta)}) = f^*_{\mathcal{O}}(v_A(\hat{P}_{A\in\Delta})) \tag{6.6}$$

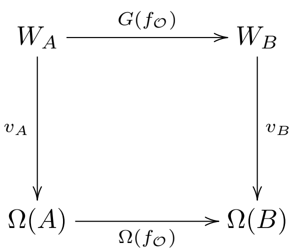
Substituting equation 6.5 in equation 6.6 we have:

$$v_B(G(f_{\mathcal{O}})(\hat{P}_{A\in\Delta})) = f^*_{\mathcal{O}}(v_A(\hat{P}_{A\in\Delta})) = \Omega(f_{\mathcal{O}})(v_A(\hat{P}_{A\in\Delta})) \tag{6.7}$$

where the last equality in equation 6.7 holds by definition of a pullback of a sieve.

Diagrammatically, equation 6.7 can be expressed as

Diagram 6.1



Which is precisely the defining condition of a natural transformation (Definition 5.6) between the two presheaves G and Ω. This leads directly to the next definition.

Definition 6.8 *Given a generalized valuation v on $P(\mathcal{H})$ there exists a corresponding **natural transformation** $N^v : G \to \Omega$ such that for any element $\hat{A} \in \mathcal{O}$ the following relation holds*

$$N^v_A(\hat{P}) := v_A(\hat{P})$$

$\forall(\hat{P}) \in W_A = G(\hat{A})$. *Where* $N^v_A(\hat{P}) := \{f : B \to A | G(f_{\mathcal{O}})(\hat{P}) \in W_B\}$.

The existence of the natural transformation N^v, entails a one-to-one correspondence between subobjects of G and valuation functions, i.e. each generalized valuation v on $P(\mathcal{H})$ corresponds to a unique subobject of G.

Global Section and Valuations

Contrary to what said in Section 5.5.2 it is possible to define generalized valuations in terms of global sections of a certain presheaf. In fact, since elements of exponentials Ω^G (Definition 8.5) are in one to one correspondence with morphisms from G to Ω (Definition 8.6) we can then define a valuation as a global section of the presheaf Ω^G.

6.3 Justifying the choice of Generalised Valuation in Quantum Theory

Up to now, from the Quantum Formalism and the Kochen-Specker Theorem we have deduced the following:

- Valuations have to be contextual.
- Truth values have to be multi-valued.
- Even if truth values of propositions of the form "$A \in \Delta$" can not be assessed to be either true or false, one can nevertheless assign truth values to coarse granings of "$A \in \Delta$", i.e. propositions of the form "$f(A) \in f(\Delta)$".

- The logic of propositions being evaluated is related in some way to the logic of the space of truth values. This implies that the space of truth values is equipped with some kind of logic.

- Valuations are such that they identify truth values with subobjects of G and, therefore, induce a one to one correspondence between natural transformations and subobjects of G.

- A definition of generalised valuation is required, which enables us to define FUNC in such a way, so that it does not fall in the Kochen-Specker " no-go" Theorem.

A sieved-valued valuation satisfies all the above requirements. In fact, we have:

1. *Contextuality.* The property of truth values being contextual arises naturally, from identifying truth values of propositions of the form "$A \in \Delta$" with sieves of coarse grainings of "$A \in \Delta$" defined on each operator $\hat{A}$.

   Let us consider a pair of self-adjoint operators $\hat{A}$ and $\hat{B}$ such that:

   a) $[\hat{A}\hat{B}] \neq 0$.

   b) Their spectral decomposition contains a common spectral projector $\hat{P}$ i.e $\hat{A} = \sum_{m=1}^{M} a_m \hat{P}_m$ and $\hat{B} = \sum_{n=1}^{N} b_n \hat{P}_n$.

   Given these conditions it follows that one can either write $\hat{P}$ in terms of $\hat{A}$, or in terms of $\hat{B}$, leading to two different propositions (Section 4.2) $(A \in \Delta)$ and $(B \in \Delta_1)$ with same projector, i.e. $\hat{P} = \hat{E}[A \in \Delta] = \hat{P} = \hat{E}[B \in \Delta_1]$.

   The truth value of these propositions given by the sieved-valued valuation will not be the same, since the sieve on $\hat{B}$ is different from the sieve on $\hat{A}$, therefore the truth value of a proposition will differ according to the context ("stage of truth") in which we decide to evaluate it, i.e. according to which operator we choose to express that proposition. Specifically, the truth value of $(A \in \Delta)$ is $v(A \in \Delta) \in \Omega(A)$, while the truth value of $(B \in \Delta_1)$ is $v(B \in \Delta_1) \in \Omega(B)$.

Moreover, the algebra $\Omega(A)$ to which $v(A \in \Delta)$ belongs, represents the Heyting algebra formed by the set of sieves on $\hat{A}$ therefore, it depends on the context $\hat{A}$.

2. *Multi-valued truth values.* By identifying truth values with set of sieves, truth values become multi-valued.

3. *Relation between logic of proposition being evaluated and logic of the space of truth values.* Since the set of sieves on each operator associated with a Quantum proposition forms an Heyting algebra, it is possible to define logical connectives in terms of set relations of the subsets in each Heyting algebra. This enables us to map the logical structure of Quantum propositions to the logical structure of the space of truth values (Definition 6.1).

4. *Truth values of propositions are in one to one correspondence with subobjects of G.* This condition arises naturally from definition of Ω being a subobject classifier of G (Section 1.9.3).

5. *FUNC.* In order to prove that a sieved valued valuation respects the condition FUNC we need to consider the following theorem [1]:

   Theorem 6.1 *Given a sieved-valued valuation v on G, then the condition FUNC is respected iff, $\forall \hat{A}$ the function $N^v : G \rightarrow \Omega$ defined by:*

$$N^v_A(\hat{P}) = v_A(\hat{P})$$

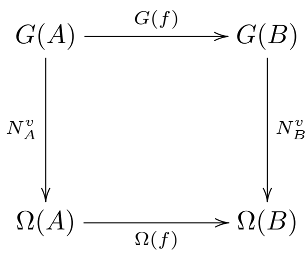
represents a natural transformation from G to Ω.

Proof 6.5 *Let us suppose that:*

$$N^v_A(\hat{P}) = v_A(\hat{P}) \tag{6.8}$$

Given a function $f : \hat{B} \rightarrow \hat{A}$ we get the following commuting diagram:

Diagram 6.2



which entails :

$$(N^v_B o G(f))(\hat{P}) = (\Omega(f) o N^v_A)(\hat{P}) \tag{6.9}$$

substituting condition 6.8 *into equation* 6.9 *we get:*

$$v_B(G(f)(\hat{P})) = \Omega(f)(v_A(\hat{P}))$$

which is precisely the FUNC condition.

Given this theorem, the FUNC condition is a necessary condition for sieved valued valuations to give rise to natural transformations from the presheaf G to the subobject classifier Ω. To elaborate, if one were to choose a sieved valued valuation, then the FUNC condition would be automatically satisfied.

6.4 Relation between Partial and General Valuation

Now that we have defined partial and generalized valuations it is worth analysing how these two valuations relate to each other and, whether it is possible to derive one from the other. Keeping in mind the definition of partial valuation V, it is possible to extend the domain of applicability of V in the following way. Consider an operator $\hat{A}$ such that $\hat{A} \notin domV$ for a given Δ, then it is possible that a function $f : \sigma(A) \to \mathbb{R}$ exists such that:

1. $f(\hat{A}) \in domV$
2. $V(f(\hat{A})) = f(\Delta)$

This is possible because, as shown in previous sections, the proposition $f(A) \in f(\Delta)$ represents the coarse graining of the proposition $A \in \Delta$, therefore $\hat{P}_{A\in\Delta} \leq \hat{P}_{f(A)\in f(\Delta)}$ in the lattice $P(\mathcal{H})$. This reasoning leads to the following definition.

Definition 6.9 *Given a partial valuation V defined on* $domV \subseteq \mathcal{O}_d$ *then the associated generalised valuation defined on propositions of the form* $A \in \Delta \;\; \forall \hat{A} \in \mathcal{O}_d$ *is:*

$$v^V(A \in \Delta) := \{f_{\mathcal{O}_d} : \hat{B} \to \hat{A} | \hat{B} \in domV, V(\hat{B}) \in f(\Delta)\} \tag{6.10}$$

The conceptual meaning behind equation 6.10, is that the truth value of propositions $A \in \Delta$ at the stage of truth $\hat{A}$, is given in terms of the coarse graining propositions $f(A) \in f(\Delta)$, which are evaluated as being totally true at their own stage of truth $f(A)$. In order for v^V, as defined by 6.10 to represent, indeed, a generalised valuation, we need to prove the following conditions.

- $v^V(A \in \Delta)$ **defines a sieve on** $\hat{A}$

Proof 6.6 *Let us suppose that* $f_{\mathcal{O}_d} : \hat{B} \rightarrow \hat{A} \in v^V(A \in \Delta)$*, i.e.* $\hat{B} \in domV$ *and* $V(B) = f(\Delta)$ *then, given a morphism* $g_{\mathcal{O}_d} : \hat{C} \rightarrow \hat{B}$ *such that* $\hat{C} = g_{\mathcal{O}_d}(\hat{B})$*, the following conditions hold:*

a) $\hat{C} \in domV$

b) $V(\hat{C}) = V(g_{\mathcal{O}_d}(\hat{B})) = V(g_{\mathcal{O}_d} o f_{\mathcal{O}_d})(\hat{A}) = (g_{\mathcal{O}_d} o f_{\mathcal{O}_d})(V(\hat{A})) = h_{\mathcal{O}_d}(V(\hat{A}))$

where $h_{\mathcal{O}_d} = g_{\mathcal{O}_d} o f_{\mathcal{O}_d} : \hat{C} \rightarrow \hat{A}$.

Note that the second and third equality are a consequence of condition (a).

Since $V(\hat{A}) \in \Delta$ *it follows, from (b), that* $h_{\mathcal{O}_d}(V(\hat{A})) \in h_{\mathcal{O}_d}(\Delta)$ *therefore* $h_{\mathcal{O}_d} \in v^V(A \in \Delta)$ *i.e.* $v^V(A \in \Delta)$ *is a sieve on* $\hat{A}$

- **Functional Composition Principle**:

  $v^V(C \in h(\Delta)) = h^*_{\mathcal{O}_d}(v^V(A \in \Delta))$

  with $h_{\mathcal{O}_d} : \hat{C} \rightarrow \hat{A}$.

Proof 6.7 *Let us consider*

$$v^V(C \in h(\Delta)) := \{k_{\mathcal{O}_d} : \hat{D} \rightarrow \hat{C} | \hat{D} \in domV, V(\hat{D}) = k(h(\Delta))\} \quad (6.11)$$

then $(h_{\mathcal{O}_d} o k_{\mathcal{O}_d}) : \hat{D} \rightarrow \hat{A} \in v^V(A \in \Delta)$ *by definition of equation* 6.11*, therefore we can rewrite equation* 6.11 *as*

$$v^V(C \in h(\Delta)) := \{k_{\mathcal{O}_d} : \hat{D} \rightarrow \hat{C} | h_{\mathcal{O}_d} o k_{\mathcal{O}_d} \in v^V(A \in \Delta)\} \quad (6.12)$$

Comparing the above equation with equation 5.2*, it is easy to see that equation* 6.12 *is the definition of a pullback of a sieve, therefore* $v^V(C \in h(\Delta)) = h^*_{\mathcal{O}_d}(v^V(A \in \Delta))$.

- **Null Proposition Condition**:

  $v^V_A(\hat{0}) = 0_A$

  where 0_A represents the empty sieve on $\hat{A}$

Proof 6.8 *The null propositions corresponds to propositions of the form "$A \in \emptyset$" and their value is:*

$$v^V(A \in \emptyset) := \{f_{\mathcal{O}_d} : \hat{B} \to \hat{A} | \hat{B} \in domV, V(\hat{B}) \in f(\emptyset)\}$$

But, since $f(\emptyset) = \emptyset$ *then,* $v^V(A \in \emptyset) = \emptyset$*, where* $\emptyset$ *represents the zero element of the Heyting algebra* $\Omega(\hat{A})$*. Therefore* $v^V_A(\hat{0}) = 0_A$*.*

- **Monotonicity Condition**:

$$A \in \Delta_1 \leq A \in \Delta_2 \Rightarrow v^V(A \in \Delta_1) \leq v^V(A \in \Delta_2)$$

Proof 6.9 *Let us consider two propositions* $A \in \Delta_1$ *and* $A \in \Delta_2$ *such that* $A \in \Delta_1 \leq A \in \Delta_2$*, i.e.* $\Delta_1 \subseteq \Delta_2$*. This implies that* $f(\Delta_1) \subseteq f(\Delta_2)$*, therefore the relation between the two valuations would be:*

$$v^V(A \in \Delta_1) = \{f_{\mathcal{O}_d} : \hat{B} \to \hat{A} | \hat{B} \in domV, V(\hat{B} \in f(\Delta_1)\}$$

$$\leq v^V(A \in \Delta_2) = \{f_{\mathcal{O}_d} : \hat{B} \to \hat{A} | \hat{B} \in domV, V(\hat{B} \in f(\Delta_2)\}$$

therefore $v^V(A \in \Delta_1) \leq v^V(A \in \Delta_2)$*.*

- **Exclusivity Condition**:

$$(A \in \Delta_1) \wedge (A \in \Delta_2) = \emptyset \ \text{ and } \ v^V(A \in \Delta_1) = true_A \downarrow \hat{A} \quad \Rightarrow \quad v^V(A \in \Delta_2) < true_A$$

Proof 6.10 *Let us consider two propositions* $A \in \Delta_1$ *and* $A \in \Delta_2$ *such that* $(A \in \Delta_1) \wedge (A \in \Delta_2) = \emptyset$*. Then, it follows that* $\Delta_1 \cap \Delta_2 = \emptyset$*.*

Let us also suppose also that $v^V(A \in \Delta_1) = true_A = \downarrow \hat{A}$*, (i.e.* $id_A \in v^V(A \in \Delta_1)$*), then it follows that* $A \in domV$ *and* $V(\hat{A}) \in \Delta_1$*. Therefore,* $V(\hat{A}) \notin \Delta_2$ *which entails that* $id_A \notin v^V(A \in \Delta_1)$*, then* $v^V(A \in \Delta_2) < true_A$*.*

- **Unit Proposition Condition**

$$(v^V(A \in \sigma(\hat{A})) = true_A) \equiv (v^V_A(\hat{1}) = true_A)$$

Proof 6.11 *Keeping in mind that for discrete spectrum* $f(\sigma(\hat{A})) = \sigma(f(\hat{A}))0\sigma(\hat{B})$ *which means that* $V(B) \in f(\sigma(\hat{A}))$ *always, the generalized valuation* v^V *of* $(A \in \sigma(\hat{A}))$ *is*

$$v^V(A \in \sigma(\hat{A})) = \{f_{\mathcal{O}_d} : \hat{B} \to \hat{A} | \hat{B} \in domV, V(\hat{B}) \in f(\sigma(\hat{A}))\}$$
$$= \{f_{\mathcal{O}_d} : \hat{B} \to \hat{A} | \hat{B} \in domV\}$$

which is equivalent to

$$v^V(A \in \sigma(\hat{A})) = domV \cap \downarrow \hat{A} \quad \subseteq \downarrow \hat{A} \tag{6.13}$$

iff $\hat{A} \notin domV$*, therefore we have:*

$$(v^V(A \in \sigma(\hat{A})) \neq true_A) \equiv (v^V_A(\hat{1}) \neq true_A)$$

We have see that the unit proposition condition is not satisfied. This means that propositions such as *"A has some value"* are not evaluated as being totally true. A consequence of equation 6.13 is that the higher the rate of truth of a proposition "$A \in \Delta$", the greater is the sieve $v^V(A \in \Delta)$ defined on $\hat{A}$. In other words, the closer the operator $\hat{A}$ is to dom V, the more truthful the proposition "$A \in \Delta$" becomes.

6.5 Examples of Truth Values in Quantum Theory

Now that we have defined what is meant by a generalized valuation of a Quantum Proposition, it is worth analysing some concrete examples of generalized valuations that arise in Quantum Theory. Given a quantum state $|\Psi\rangle \in \mathcal{H}$, we have the following definition for the associated generalized valuation.

Definition 6.10 *Given a vector $|\Psi\rangle \in \mathcal{H}$, the associated* ***generalised valuation*** $v^{|\Psi\rangle}$ *is*

$$v^{|\Psi\rangle}(A \in \Delta) := \{f_{\mathcal{O}} : \hat{B} \to \hat{A} | \hat{E}[B \in f(\Delta)]\Psi = \Psi\} \tag{6.14}$$

$$:= \{f_{\mathcal{O}} : \hat{B} \to \hat{A} | \hat{P}_{B \in f(\Delta)}\Psi = \Psi\} \tag{6.15}$$

where Δ is a Borel subset of $\sigma(\hat{A})$.

We should note that, when Ψ is an eigenstate of $\hat{A}$, then $v^{|\Psi\rangle}(A \in \Delta) = \downarrow \hat{A}$. We now have to show that $v^{|\Psi\rangle}$, as defined above, satisfies the required conditions for being a proper generalized valuation.

- $v^{|\Psi\rangle}(A \in \Delta)$ **is a sieve on** $\hat{A}$

  Proof 6.12 *Let us consider a function $f_{\mathcal{O}} \in v^{|\Psi\rangle}(A \in \Delta)$, from equation* 6.14 *it follows that $\hat{E}[B \in f(\Delta)]\Psi = \Psi$. Given a function $g_{\mathcal{O}} : \hat{C} \to \hat{B}$ such that $\hat{C} = g(\hat{B})$, we then have:*

$$\begin{aligned} \hat{E}[g(f(A)) \in g(f(\Delta))]\Psi &= \hat{E}[C \in g(f(\Delta))]\Psi \\ &= \hat{E}[g^{-1}(C) \in f(\Delta)]\Psi \\ &= \hat{E}[B \in f(\Delta)]\Psi \\ &= \Psi \end{aligned}$$

  therefore $f_{\mathcal{O}} o g_{\mathcal{O}} \in v^{|\Psi\rangle}(A \in \Delta)$

- **Null Proposition Condition**

$$v^{|\Psi\rangle}(\hat{O}) = v^{|\Psi\rangle}(A \in \emptyset) = 0_A$$

Proof 6.13

$$v^{|\Psi\rangle}(A \in \emptyset) = \{f_{\mathcal{O}} : \hat{B} \to \hat{A} | \hat{E}[B \in f(\emptyset)]\Psi = \Psi\}$$

$$= \{f_{\mathcal{O}} : \hat{B} \to \hat{A} | \hat{0}\Psi = \Psi\}$$

$$= \emptyset = 0_A$$

- **Functional Composition Condition**

$$v^{|\Psi\rangle}(f(A) \in f(\Delta)) = f^*_{\mathcal{O}}(v^{|\Psi\rangle}(A \in \Delta))$$

Proof 6.14

$$v^{|\Psi\rangle}(f(A) \in f(\Delta)) = \{g_{\mathcal{O}} : \hat{C} \to \hat{B} | \hat{E}[C \in f(g(\Delta))]\Psi = \Psi\}$$

$$= \{g_{\mathcal{O}} : \hat{C} \to \hat{B} | \hat{E}[C \in h(\Delta)]\Psi = \Psi\}$$

$$= \{g_{\mathcal{O}} : \hat{C} \to \hat{B} | h_{\mathcal{O}} \in v^{|\Psi\rangle}(A \in \Delta)\}$$

$$= f^*_{\mathcal{O}}(A \in \Delta)$$

where $h_{\mathcal{O}} = f_{\mathcal{O}} o g_{\mathcal{O}} : \hat{C} \to \hat{A}$

- **Monotonicity Condition**

$$\Delta_1 \subseteq \Delta_2 \Rightarrow v^{|\Psi\rangle}(A \in \Delta_1) \leq v^{|\Psi\rangle}(A \in \Delta_2)$$

Proof 6.15 *Let us suppose that* $\Delta_1 \subseteq \Delta_2$*, then we have* $\hat{E}[A \in \Delta_1] \leq \hat{E}[A \in \Delta_2]$*. This entails the following relation between the two valuations:*

$$v^{|\Psi\rangle}(A \in \Delta_1) := \{f_{\mathcal{O}} : \hat{B} \to \hat{B} | \hat{E}[B \in f(\Delta_1)]\Psi = \Psi\}$$

$$\leq v^{|\Psi\rangle}(A \in \Delta_2) := \{f_{\mathcal{O}} : \hat{B} \to \hat{B} | \hat{E}[B \in f(\Delta_2)]\Psi = \Psi\}$$

- **Exclusivity Condition**

$$\Delta_1 \cap \Delta_2 = \emptyset, v^{|\Psi\rangle}(A \in \Delta_1) = \downarrow \hat{A} \quad \Rightarrow \quad v^{|\Psi\rangle}(A \in \Delta_2) < \downarrow \hat{A}$$

Proof 6.16 *Consider* $v^{|\Psi\rangle}(A \in \Delta_1) =\downarrow \hat{A}$. *Then* $id_A : \hat{A} \rightarrow \hat{A} \in v^{|\Psi\rangle}(A \in \Delta_1)$. *This implies that* $\hat{E}[id_A(A) \in id_A(\Delta_2)]\Psi = 0$, *therefore* $id_A \notin v^{|\Psi\rangle}(A \in \Delta_2)$, *i.e.* $v^{|\Psi\rangle}(A \in \Delta_2) <\downarrow \hat{A}$

We should now check if the unity proposition condition is satisfied.

- **Unity Proposition Condition**

$$v^{|\Psi\rangle}(A \in \sigma(\hat{A})) =\downarrow \hat{A}$$

Proof 6.17

$$v^{|\Psi\rangle}(A \in \sigma(\hat{A})) := \{f_{\mathcal{O}} : \hat{B} \rightarrow \hat{A} | \hat{E}[B \in f(\sigma(\hat{A}))]\Psi = \Psi\} \tag{6.16}$$

From equation 6.2 *it can be seen that* $\hat{E}[A \in \sigma(\hat{A})] = \hat{1}$. *However, since* $\hat{E}[A \in \sigma(\hat{A})] \leq \hat{E}[B \in f(\sigma(\hat{A}))]$, *it follows that* $\hat{E}[B \in f(\sigma(\hat{A}))] = \hat{1}$. *Therefore equation* 6.16 *becomes* $v^{|\Psi\rangle}(A \in \sigma(\hat{A})) := \{f_{\mathcal{O}} : \hat{B} \rightarrow \hat{A} | \hat{1}\Psi = \Psi\}$, *i.e.* $v^{|\Psi\rangle}(A \in \sigma(\hat{A})) =\downarrow \hat{A}$.

We have then proven that, in contrast to v^V, $v^{|\Psi\rangle}$ does satisfy the unity condition.

It is worth, at this point to mention the negation operation for $v^{|\Psi\rangle}$.

Negation Operation

The negation operation for the generalised valuation $v^{|\Psi\rangle}(A \in \Delta)$ as defined in 6.14 is:

$$\neg v^{|\Psi\rangle}(A \in \Delta) := \{f_{\mathcal{O}} : \hat{B} \rightarrow \hat{A} | \forall g_{\mathcal{O}} \hat{C} \rightarrow \hat{B}, f_{\mathcal{O}} o g_{\mathcal{O}} \notin v^{|\Psi\rangle}(A \in \Delta)\}$$

From the above equation it is clear that the negation operation differs radically according to whether we choose the category $\mathcal{O}$ or $\mathcal{O}^*$. In fact if we chose $\mathcal{O}$ then the negation would always correspond to the empty set since the inclusion of the unit operator $\hat{1}$ as a possible stage of truth implies that, for every function $f_{\mathcal{O}}$, there exists a function $h_{\mathcal{O}} : \hat{1} \rightarrow \hat{B}$ such that $f_{\mathcal{O}} o h_{\mathcal{O}} = 1_{\hat{A}}$ but $1_{\hat{A}} \in v^{|\Psi\rangle}(A \in \Delta)$, therefore $\neg v^{|\Psi\rangle}(A \in \Delta) = \emptyset$. If instead we used the category $\mathcal{O}^*$ this would not happen.

Chapter 7

Conclusion

Any attempt to give a realist interpretation of Quantum Theory gets ruled out by the Kochen-Specker Theorem and the Bell inequalities. Nevertheless, by using Topos theory and in particular the logic of presheaves, it is possible to define a sieved-valued valuation for Quantum proposition, such that the truth values it assigns to Quantum propositions are multi-valued and contextual. This features (in a Quantum valuation) are desirable since they agree we the mathematical formalism of Quantum Theory. Moreover adopting the logic of presheaves it is possible to uniquely define the logical connectives in such a way the metalanguage/object-language distinction is not violate. Another advantage of presheaf logic is that it is a Heyting algebra, and therefore, differently from other Quantum logic, it is distributive. This implies that it can be used as a deductive system of reasoning.

Since Topos theory revealed itself as an appropriate tool to re-define the Kochen-Specker theorem, could it then be used to re-define the Bell inequalities?. This is an interesting question which probes at the hart of the meaning of probabilities in Quantum theory. In fact, the essential ingredient in deriving the Bell inequalities is the probability distribution map, $\rho : P(\mathcal{H}) \rightarrow \{0,1\}$ from the lattice of projection operators to the Boolean algebra $\{0,1\}$, such that, given an operator $\hat{A}$ we have $\rho(\hat{A}) = Prob(A = a_m; |\psi\rangle)$. Therefore, if a measurement on $\hat{A}$ is made, the probability of obtaining the result a_m is $\rho(\hat{A})$. Within this frame work, probabilities are interpreted in a statistical way, i.e. they refer to the *relative-frequency* of obtaining different results when an observable is measured on a large number of "identically-prepared" systems. Therefore properties are not possessed by a system but they are to be interpreted only as a result of measurement. As Bell puts it, the verb "to be" becomes "to be measured". In order to re-define the probability distribution map in terms of topos theory, we would have to adopt a different conception of probabilities. In fact we would have to abandon the Boolean algebras $\{0,1\}$ and adopt a Heyting algebra as the algebra in which probabilities take there values. This would entail that the "relative-frequency" interpretation of probabilities would no longer hold. An alternative would be to consider probabilities as "potentiality" for different results, adopting in such a way the concept of latent properties put forward by Bhom. Then again we would have to define what it means to attribute latent properties to a system.

All this are intriguing questions that would generate a deeper insight in the understanding of Quantum Theory and the view of the world it proposes.

Chapter 8

Appendix

8.1 Elements in a Category

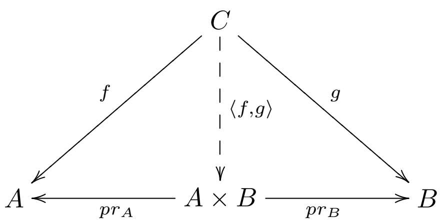
Definition 8.1 *A* terminal object *in a category* $\mathscr{C}$ *is a* $\mathscr{C}$*-object 1 such that, given any other* $\mathscr{C}$*-object* A*, there exists one and only one* $\mathscr{C}$*-arrow from* A *to 1.*

Definition 8.2 *An* initial object *in a category* $\mathscr{C}$ *is a* $\mathscr{C}$*-object 0 such that, for every other* $\mathscr{C}$*-object* A*, there exists one and only one* $\mathscr{C}$*-arrow from 0 to* A*.*

Definition 8.3 *A product of two objects* A *and* B *in a category* $\mathscr{C}$ *is a third* $\mathscr{C}$*-object* $A \times B$ *together with a pair of* $\mathscr{C}$*-projection arrows:*

$pr_A : A \times B \to A$ *and* $pr_B : A \times B \to B$

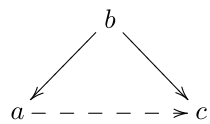
such that, given any other pair of $\mathscr{C}$*-arrows* $f : C \to A$ *and* $g : C \to B$*, there exists a unique arrow* $\langle f, g \rangle : C \to A \times B$ *such that the following diagram* *commutes*



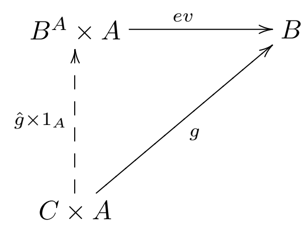
i.e.

$$pr_A o\langle f,g\rangle = f \quad and \quad pr_b o\langle f,g\rangle = g$$

Definition 8.4 *Given any $\mathscr{C}$-object a and a $\mathscr{C}$-arrow $f : b \to a$, we say that f is a* co-limit *of $\mathscr{C}$ iff given any other $\mathscr{C}$-arrow $g : b \to c$ there exists one and only one $\mathscr{C}$-arrow $h : a \to c$ such that the following diagram commutes:*

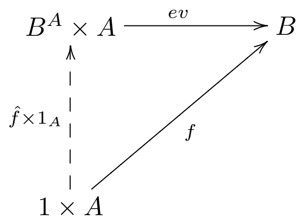


Definition 8.5 *An* exponentiation *from a $\mathscr{C}$-object A to a $\mathscr{C}$-object B is a maps $f : A \to B$ denoted B^A together with an evaluation map $ev : B^A \times A \to B$, with the property that given any other $\mathscr{C}$-object C and $\mathscr{C}$-arrow $g : C \times A \to B$ there exists a unique arrow $\hat{g} : C \to B^A$ such that the following diagram commutes*



Definition 8.6 *objects of B^A are in one-to-one correspondence with maps of*

the form $A \to B$. To see this let us consider the following commuting diagram



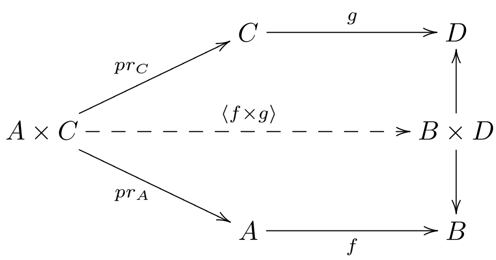
where $f : 1 \times A \to B$ is unique. But $1 \times A \equiv A$ therefore to each element of B^A there corresponds a unique function $A \to B$.

8.2 Arrows in a Category

Definition 8.7 *an arrow $f : A \to B$ in a category $\mathscr{C}$ is said to be monic (left cancelable) iff given any other pair of $\mathscr{C}$-arrows $g, h : C \to A$ then,*

$$\textit{whenever} \quad fog = foh \quad \textit{then} \quad g = h$$

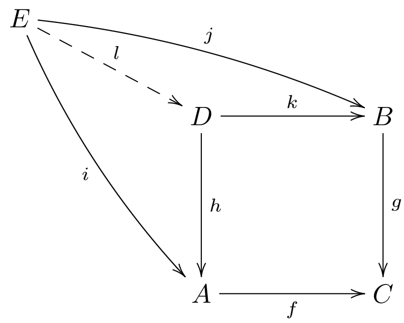
Definition 8.8 *Given a pair of $\mathscr{C}$-arrows $f : A \to B$ and $g : C \to D$ then, the product arrow $f \times g : A \times C \to B \times D$ is the unique $\mathscr{C}$-arrows $\langle fopr_A, gopr_C \rangle = \langle f \times g \rangle$ where pr_A and pr_C are the projection maps. Diagrammatically we have:*



Definition 8.9 *a pullback of a pair of functions $f : A \to B$ and $g : B \to C$ in a category $\mathscr{C}$ is a pair of $\mathscr{C}$-arrows $h : D \to A$ and $k : D \to B$, such that the following conditions are satisfied:*

1. *$foh = gok$*

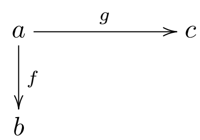
2. *given two functions $i : E \to A$ and $j : E \to B$, where $foi = goj$ then, there exists a unique $\mathscr{C}$-arrow l from E to D such that the outer rectangle of the following diagram commutes*



i.e.

$$i = hol \qquad j = kol$$

Definition 8.10 *A pushout of two arrows $b \xleftarrow{f} a \xrightarrow{g} c$ with common domain, is defined as the co-limit of the following diagram*



Definition 8.11 *Given a subset $A \subset D$, then the characteristic function χ_A is defined as follows*

$$\chi_A(x) = \begin{cases} 1 \;\; if \;\; x \in A \\ \\ 0 \;\; if \;\; x \notin A \end{cases} \tag{8.1}$$

i.e. for those elements of D in A χ_A gives out put 1 for all other elements it gives output 0.

8.3 Sets

Definition 8.12 *a* pre-ordered *set is a set with the property that, between any two objects there is at most one arrow. This entails that there exists a binary relation R between the objects of the pre-ordered set such that the following holds:*

1. *aRa (reflexivity)*
2. *if aRb and bRc then aRc (transitivity)*

Definition 8.13 *a* poset *is a pre-ordered set with the extra property of being antisymmetric:* $pRq \quad and \quad qRp \quad \Rightarrow p = q$

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