Is it True; or is it False; or Somewhere In Between? The Logic of Quantum Theory[1]

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Abstract

The paper contains a relatively non-technical summary of some recent work by the author and Jeremy Butterfield. The goal is to find a way of assigning meaningful truth values to propositions in quantum theory: something that is not possible in the normal, instrumentalist interpretation. The key mathematical tool is presheaf theory where, multi-valued, contextual truth values arise naturally. We show how this can be applied to quantum theory, with the 'contexts' chosen to be Boolean subalgebras of the set of all projection operators.

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1 What is Quantum Theory About?

Consider the following two statements concerning a physical quantity A and a real number a. The critical words are italicised.

> "*If* a *measurement* of A is made, the probability that the *result* will be a is p."

> "The quantity A *has* a value, and the probability that this value *is* a is p."

The first statement is an instrumentalist way of talking about physics: it does not concern itself with what 'is the case' but only with the results of measurements. The essential counterfactuality is captured by the opening 'If': the statement asserts what would happen (or, more precisely, the probability of what would happen) *if* a certain action is taken. It is silent about the situation in which no measurement is made.

The second statement is very different. It reflects a typical realist view of the world in which, at any moment of time, any physical quantity is deemed to *possess* a value, even if we do not know what that value is. Concomitantly, any proposition asserted about the values of physical quantities is either true or false: a nice, simple, black-or-white view of the world.

In classical physics (and, indeed, in the normal, 'commonsense' world) no fundamental distinction between these statements need be made. If some one asks "Why did the measurement of the physical quantity A give the particular result that it did?", the obvious answer is that A *possessed* that value at the time the measurement was made. A good measurement simply reveals 'what is the case'.

However, the situation in quantum physics is radically different. The standard interpretation of the theory is unashamedly instrumentalist: indeed, many proponents would insist that it is generally meaningless to even talk about the values of physical quantities other than in the counterfactual language of measurement results.

Quantum theory is usually taught in this way and, of course, within its own limitations the interpretation works extremely well. The rapid growth of the solid-state industries is a striking demonstration of this, as are the activities of the particle physicists at CERN, and those of a host of other scientists and engineers who use quantum theory on a daily basis. However, many scientists (and non-scientists too) feel compelled to seek a deeper reality that lies beneath such an instrumentalist veneer; and even in a strict instrumentalist framework there is still the infamous 'measurement problem' that arises when one probes more deeply into the question of what type of interaction should count as a 'measurement'.

The desire to develop a more realist interpretation of quantum theory reaches an apotheosis in the context of quantum cosmology: the application of quantum theory to the universe itself. However finding such an interpretation is not an easy task, not least because of the difficulty in specifying what is really meant by 'realism' and a 'realist' interpretation. This is, of course, a huge philosophical issue, but in the context of the physical sciences one can tentatively say that a realist interpretation is one in which (i) propositions about the physical world are handled using standard Boolean logic; and (ii) at any moment of time, each such proposition is either true or false. The underlying assumption is that, at any time, every physical quantity *possesses* a definite value. Propositions about the system are then statements that each member of some set of physical quantities has a value that lies in a specific range.

In classical physics, the collection of all propositions about a physical system does indeed form a Boolean algebra (see Section 2); and, for each state of the system, any proposition about the system is indeed either true or false. Of course, all this is in accord with our ordinary, commonsense view of the world.

However, in quantum theory the situation is very different. For example, consider a simple system with a two-dimensional vector space of states, and with the state vector $|\psi\rangle$ shown in Figure 1. This could represent the spin degrees of freedom of an electron, with the '$\uparrow$, $\downarrow$' symbols corresponding to the z-component of spin, S_z, being $+\frac{1}{2}\hbar$ and $-\frac{1}{2}\hbar$ respectively.

In the conventional interpretation of quantum theory, all that can be said about the value of S_z is that *if* a measurement is made of the z-component of spin, then the *probabilities* of getting the results $-\frac{1}{2}\hbar$ ('down') and $+\frac{1}{2}\hbar$ ('up') are $\cos^2\theta$ and $\sin^2\theta$ respectively (for simplicity I have taken a real, rather than complex, vector space). However, unless $\theta = 0^0$ or 90^0 (so that $|\psi\rangle$ is then an eigenvector of $\hat{S}_z$) nothing can be said about the *value* of the spin: *i.e.*, it cannot be asserted meaningfully that the spin has/possesses any specific value. In particular, the proposition "the electron has spin down" (or spin up) cannot be assigned a meaningful truth value.

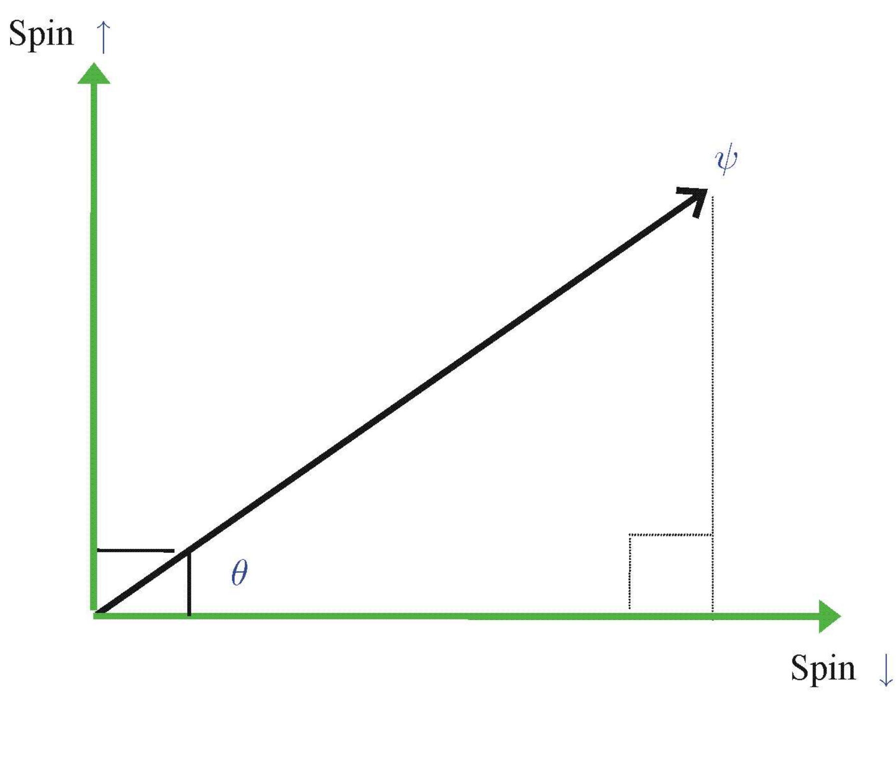

Figure 1: The quantum state $|\psi\rangle$ is a non-trivial superposition of the two eigenstates ('spin up' and 'spin down') of $\hat{S}_z$. As a consequence, the proposition "the electron has spin down" has no meaningful truth value.

This inability to sustain a simple realist interpretation of the theory is not just some whimsical psychological preference of the quantum physicist. Rather, it is an inevitable consequence of the famous Kochen-Specker theorem [2]. This asserts the nonexistence[3] of valuations[4] in quantum theory, subject only to the rather plausible requirement that the value of a function of a physical quantity should be the result of applying that function to the value of the quantity. In symbols, if V is a putative value function, and if f is a real-valued function of real numbers then, if A is any physical quantity, the requirement is

$$V(f(A)) = f(V(A)). \tag{1.1}$$

For example, the value of the quantity 'energy-squared' could reasonably be expected to be the square of the value of the energy.

The Kochen-Specker is a major result in quantum theory, and is the motivational force behind the present paper. When applied to propositions, the theorem asserts the non-existence of any consistent assignment of true-false values to the propositions in quantum theory.

One common response to the Kochen-Specker theorem is to note that although it forbids any absolute assignment of truth values, it does not exclude ones that are *contextual*. Here, 'contextual' means that the truth value given to a proposition depends on which other compatible (meaning 'simultaneously measurable') propositions are given values at the same time. Of course, this does not say how such a contextual valuation might be obtained, or what properties it should possess. The aim of the present paper is to show how one particular such scheme is already contained within the existing formalism of quantum theory, without the need to add hidden variables, or the like. However this scheme has the feature that, as well as being contextual, the truth values are also *multi-valued*. We shall refer to truth values that are multi-valued and/or contextual as *generalised* truth values.

The idea of multi-valued logic has cropped up from time to time before in the history of quantum theory: for example, Reichenbach introduced the idea of a three-valued logic, so that a proposition could be true, false, or 'in between' [1]. However, a major problem in such proposed logics has always been how to define the logical operations 'and', 'or', and 'not'; in practice, the procedures have tended to be rather hit or miss. A few years ago, Jeremy Butterfield and I introduced a novel form of multi-valued logic in quantum theory that was based on the use of topos theory; or, more precisely, on the use of the special case of presheaf theory. One advantage of this new approach is that the logical operations are defined *unambiguously* by the basic mathematical structure of the relevant presheaf. It is this scheme that is described in the present paper: hopefully, in a relatively non-technical way.

The structure of the paper is as follows. In Section 2 there is a short introduction to the way logic arises in classical physics and in normal quantum theory. This includes a

[3]Actually, the theorem only holds if the dimension of the vector space of states is greater than two, whereas in the example under discussion the dimension is equal to two. However, this does not alter the general thrust of the argument being developed here.

[4]A *valuation* is a function that assigns a value (a real number) to each physical quantity. When applied to propositions, a valuation assigns a truth value: 1 for 'true', and 0 for 'false'.

demonstration of how a certain type of multi-valued logic is already present in classical physics. In Section 3 we extend this idea to quantum theory. This involves constructing a special presheaf that can be used to assign truth values that are both contextual and multi-valued. Nevertheless, the underlying logic is sufficiently like that of a Boolean algebra to enable statements about the world to be asserted and manipulated in a logical way.

2 The Logic of Physics

2.1 The Logic of Classical Physics

A key feature of classical physics is that, at any given time, the system has a definite state, and this state determines—and is uniquely determined by—the values of all the physical quantities associated with the system. The set of possible states of a system is called the 'space of states', or 'state space'. This notion of a state captures well the realist philosophy underlying classical physics.

As an example, consider a point particle moving along a line according to the laws of Newtonian physics. The state of such a system is completely determined by the values of the position, x, and momentum, p, of the particle. Thus the state space is a two-dimensional space with coordinates x and p, as shown in Figure 2.

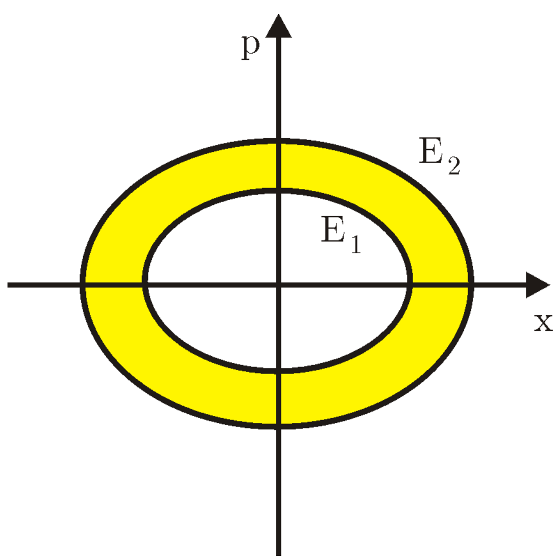

Figure 2: The classical state space of a particle moving in one dimension. The shaded area represents the set of states for a simple harmonic oscillator for which the energy E satisfies $E_1 < E < E_2$.

Of course, a point particle has physical properties other than the values of position and momentum; for example, it will have a certain energy, E. However, the energy of the particle is completely determined by its state, *i.e.*, by the values of position and momentum. For example, for a simple harmonic oscillator we have $E(x, p) = \frac{p^2}{2m} + kx^2$, where m is the mass of the particle, and k is some positive constant.

It is clear that different states can give the same value of the energy. For example, for the harmonic oscillator the set of states (x, p) for which the energy has a value E_1 is represented by the inner ellipse in Figure 2. Similarly, the outer ellipse represents the set of states for which the energy has a value E_2, with $E_1 < E_2$. Then the proposition "the energy of the system lies between E_1 and E_2" is represented by the yellow subset between the two ellipses.

This idea can be generalised to any classical system. Specifically, if $\mathcal{S}$ is the state space, then every proposition P about the system can be represented by an associated subset, $\mathcal{S}_P$, of $\mathcal{S}$: namely, the set of states for which P is true. Conversely, every subset of $\mathcal{S}$ represents a proposition. More precisely, every subset represents *many* propositions about the values of physical quantities. One sometimes says that two propositions are 'physically equivalent' if they are represented by the same subset of $\mathcal{S}$.

It is easy to see how the logical calculus of propositions arises in this picture. For suppose that P and Q are a pair of propositions, represented by the subsets $\mathcal{S}_P$ and $\mathcal{S}_Q$ respectively, and consider the proposition "P *and* Q". This is true if, and only if, both P and Q are true, and hence the subset of states representing this logical conjunction consists of those states that lie in both $\mathcal{S}_P$ and $\mathcal{S}_Q$—*i.e.*, the set-theoretic intersection $\mathcal{S}_P \cap \mathcal{S}_Q$. Thus "$P$ *and* Q" is represented by $\mathcal{S}_P \cap \mathcal{S}_Q$. Similarly, the proposition "P *or* Q" is true if either P or Q (or both) are true, and hence this logical disjunction is represented by those states that lie in $\mathcal{S}_P$ plus those states that lie in $\mathcal{S}_Q$—*i.e.*, the set-theoretic union $\mathcal{S}_P \cup \mathcal{S}_Q$. Finally, the logical negation "*not* P" is represented by all those points in $\mathcal{S}$ that do not lie in $\mathcal{S}_P$—*i.e.*, the set-theoretic complement $\mathcal{S}/\mathcal{S}_P$.

In this way, a fundamental relation is established between the logical calculus of propositions about a physical system, and the Boolean algebra of subsets of the state space. Thus the mathematical structure of classical physics is such that, *of necessity*, it reflects a realist philosophy.

However, note that the Boolean algebra structure is on the *subsets* of $\mathcal{S}$—*i.e.*, on the space of physically equivalent sets of propositions—not on the propositions themselves. For example, if P, Q are propositions, with representatives $\mathcal{S}_P$ and $\mathcal{S}_Q$ respectively, then, as explained above, the logical 'and' operation is associated with the subset $\mathcal{S}_P \cap \mathcal{S}_Q$. But this subset corresponds to *many* propositions (all physically equivalent to each other) and there is no preferred way of selecting one particular one from this set.

2.2 The Standard Logic of Quantum Theory

In quantum theory, a proposition is represented [3] by a projection operator[5] on the vector space, $\mathcal{H}$, states. Equivalently, a proposition is represented by the linear subspace, $\mathcal{H}_{\hat{P}}$ (known as the *range* of $\hat{P}$), of $\mathcal{H}$ upon which the projection operator $\hat{P}$ projects. Analogous to the situation in classical physics, many propositions can be represented by the same projection operator. As we shall see in Section 3, this has important ramifications for what we are trying to do.

If $\hat{P}$ and $\hat{Q}$ are a pair of projection operators, with corresponding subspaces $\mathcal{H}_{\hat{P}}$ and

[5] Recall that a projection operator is a hermitian operator $\hat{P}$ that satisfies $\hat{P}^2 = \hat{P}$.

$\mathcal{H}_{\hat{Q}}$ respectively, then the subspace that represents the proposition[6] "*P and Q*" is simply the intersection $\mathcal{H}_{\hat{P}} \cap \mathcal{H}_{\hat{Q}}$: we shall denote the corresponding projection operator by $\hat{P} \wedge \hat{Q}$. Similarly, the subspace that represents the proposition "*not P*" is the orthogonal complement[7] of the subspace $\mathcal{H}_{\hat{P}}$. The corresponding projection operator is $\hat{1} - \hat{P}$, where $\hat{1}$ is the unit operator.

The situation in regard to the logical 'or' operation is more complicated. Given a pair of propositions P, Q, the obvious choice to represent "*P or Q*" might seem to be the union $\mathcal{H}_{\hat{P}} \cup \mathcal{H}_{\hat{Q}}$. However, this is not a linear subspace of the vector space $\mathcal{H}$, and hence cannot represent any proposition. Instead, the proposition "*P or Q*" is represented by the *linear span* of the vectors in $\mathcal{H}_{\hat{P}} \cup \mathcal{H}_{\hat{P}}$—*i.e.*, the collection of all possible sums of vectors in $\mathcal{H}_{\hat{P}} \cup \mathcal{H}_{\hat{P}}$; the corresponding projection operator will be denoted by $\hat{P} \vee \hat{Q}$. This choice has the desirable property of *associativity*: for any three projectors $\hat{P}$, $\hat{Q}$ and $\hat{R}$ we have $\hat{P} \vee (\hat{Q} \vee \hat{R}) = (\hat{P} \vee \hat{Q}) \vee \hat{R}$. This is consonant with the logic of daily life where it is taken for granted that if P, Q, R are any three propositions, then "$(P$ *or* $Q)$ *or* R" = "P *or* $(Q$ *or* $R)$". It is easy to see that the 'and' operation is also associative: for any three projectors $\hat{P}, \hat{Q}, \hat{R}$, we have $\hat{P} \wedge (\hat{Q} \wedge \hat{R}) = (\hat{P} \wedge \hat{Q}) \wedge \hat{R}$.

However, this 'quantum logic' of projection operators differs from Boolean logic in one critical feature: it fails to be distributive. Thus, given three projectors $\hat{P}, \hat{Q}, \hat{R}$, we will generally have

$$\hat{P} \wedge (\hat{Q} \vee \hat{R}) \neq (\hat{P} \wedge Q) \vee (\hat{P} \wedge R). \tag{2.1}$$

To see how bizarre non-distributive thinking would be in daily life suppose that I was staying at a hotel and, at breakfast, the waiter said "Would you like eggs and sausage or bacon?". If I parsed this phrase as "eggs and (sausage or bacon)", I would assume that I was being offered a choice between eggs and sausage, or eggs and bacon. In other words, I would invoke the distributive law

$$E \text{ and } (S \text{ or } B) = (E \text{ and } S) \text{ or } (E \text{ and } B). \tag{2.2}$$

However, it is easy to construct a simple quantum model in which if I respond "eggs and sausage, please" I get nothing, and similarly for eggs and bacon. In fact, in this particular example, the only sensible reply to "Would you like eggs and sausage or bacon?" is "Yes please", in which case my plate would arrive with eggs plus a quantum superposition of sausage and bacon :-)

As applied to this non-distributive quantum logic, the Kochen-Specker theorem asserts the impossibility of assigning consistent true-false values to projection operators. Thus the corresponding properties cannot be said to be 'possessed' by the system. However, the Kochen-Specker theorem does not necessarily preclude the existence of truth values that are contextual and/or multi-valued, provided an appropriate mathematical structure can be found. It is to this task that we now turn.

[6] This is a rather loose way of speaking: the subspace $\mathcal{H}_{\hat{P}} \cap \mathcal{H}_{\hat{Q}}$ really represents *all* of the propositions "*P and Q*" as P and Q range over the propositions represented by $\hat{P}$ and $\hat{Q}$ respectively. Similar remarks apply to the logical 'or' and 'not' operations.

[7] The orthogonal complement of a subspace, W, of $\mathcal{H}$ is the set of all vectors that are orthogonal to every vector in W.

2.3 A Role for Multi-Valued Logic in Classical Physics

Consider first a classical system with state space $\mathcal{S}$. Each physical quantity A is represented by a real-valued function (denoted $\overline{A}$) on $\mathcal{S}$ with the interpretation that if s in $\mathcal{S}$ is a state of the system, then the value of the physical quantity A in that state is the real number $\overline{A}(s)$. As explained in Section 2.1, a proposition of the form "$A \in \Delta$" (meaning that the value of A lies in the set Δ of real numbers) is then represented by the subset $\mathcal{S}_{A\in\Delta}$ of $\mathcal{S}$ consisting of all those states s for which $\overline{A}(s)$ belongs to Δ (see Figure 3). Of course, this structure is consistent with the philosophical view that each physical quantity *has* a value for any given state of the system. In particular, any proposition asserted about the system is either true or false. Thus the proposition "$A \in \Delta$" is true if s belongs to $\mathcal{S}_{A\in\Delta}$, and it is false if it does not.

All this seems clear-cut—but is it really so? For suppose s is a state that does not belong to $\mathcal{S}_{A\in\Delta}$ but which, nevertheless, is 'almost' in this subset (so that $\overline{A}(s)$ 'almost' belongs to Δ): is there not some sense in which the proposition "$A \in \Delta$" is then 'almost true'? Contrariwise, suppose s is such that $\overline{A}(s)$ belongs to Δ, but only just so (*i.e.*, $\overline{A}(s)$ is 'close' to the edges of Δ): then is "$A \in \Delta$" not 'almost false', or 'only just true'? Such grey-scale judgements are frequently made in daily life, but there seems to be no role for them in the harsh, black-and-white mathematics of classical physics.

The situation becomes even more piquant if, rather than being given a specific (micro)-state s, we know only that s lies in some subset M (a *macro-state*) of $\mathcal{S}$.[8] What truth value, if any, can then be ascribed to the proposition "$A \in \Delta$"? If M is a subset of $\mathcal{S}_{A\in\Delta}$, it does seem correct to say that the proposition is true (perhaps even 'totally true'?), since for each state s in the macro-state M, the real number $\overline{A}(s)$ *does* belong to Δ.[9]

However, suppose M is such that there are some states s in M for which $\overline{A}(s)$ belongs to Δ, and some states for which it does not (see Figure 3). In this situation, what truth value can be given to the proposition "$A \in \Delta$"? Is there some sense in which it is 'partially true'?

As things stand, if "$A \in \Delta$" is interpreted as asserting that, for *all* s in M, the real number $\overline{A}(s)$ belongs to Δ, then the proposition is clearly false. But suppose M is 'almost' a subset of $\mathcal{S}_{A\in\Delta}$: are we not then tempted to say that the proposition "$A \in \Delta$" is 'almost true'? At the very least, in such circumstances it seems misleading to assert that the proposition is unequivocally false. And even if none of the states in M belong to $\mathcal{S}_{A\in\Delta}$ we can still imagine situations in which M is 'close' to this subset, so that it might appropriate to say that "$A \in \Delta$" is 'almost true' (or, perhaps, 'almost not false').

The difficulty in succumbing to such temptations is that the word 'almost' has no well-defined meaning in the standard mathematics that is used in physics. But what is strongly suggested by the discussion above is the need to introduce multiple truth values

[8]In statistical physics, the macro-state M would be given some probability by the theory. Of course, an assignment of probabilities to macro-states is not incompatible with a realist view in which the system *has* a definite state (and each physical quantity has a definite value) but we happen not to know what this state is, only that it lies in the subset M of $\mathcal{S}$.

[9]Although one might want to handle the situations in which M is 'only just' a subset of $\mathcal{S}_{A\in\Delta}$, so that "$A \in \Delta$" is 'almost not totally true'.

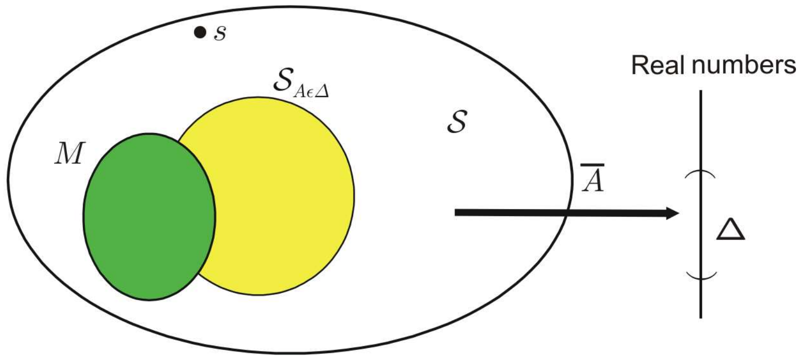

Figure 3: A diagram to aid discussing a multi-valued logic for classical macrostates. Here $\mathcal{S}$ is the state-space for the system; $M \subset \mathcal{S}$ is a macrostate; $\overline{A}$ is the real-valued function on $\mathcal{S}$ that represents a physical quantity A; and $\mathcal{S}_{A\in\Delta}$ is the subset of states s such that $\overline{A}(s)$ belongs to $\Delta \subset \mathbb{R}$.

that can interpolate between 'true' and 'false'.

2.4 The Use of Coarse-Graining

One way of introducing multi-valued logic into classical physics is to use a certain coarse-graining operation. The basic idea is rather simple. Namely, if we are in the type of situation envisaged above, where we feel reluctant to assign a simple true-false value to a proposition "$A \in \Delta$", then perhaps we can find a real-valued function f of the real numbers such that the proposition[10] "$f(A) \in f(\Delta)$" definitely *is* true. This possibility arises because the proposition "$f(A) \in f(\Delta)$" is *weaker* than the proposition "$A \in \Delta$': thus "$A \in \Delta$" *implies* "$f(A) \in f(\Delta)$" but the converse is generally not true. For example from the knowledge that a physical quantity A has the value 2, the quantity A^2 can be deduced to have the value 4. On the other hand, from the knowledge that $A^2 = 4$ all that can be deduced about the value of A is that it is equal to $+2$ or -2. This weakening of propositions can occur whenever the function f is not one-to-one.

The question now is if such weakening operations can be used to give a truth value to a proposition "$A \in \Delta$" in a macro-state M when M is not simply a subset of $\mathcal{S}_{A\in\Delta}$—*i.e.*, $\overline{A}(M)$ is not a subset of Δ. As discussed in Section 2.3, in this situation we may be reluctant to say that "$A \in \Delta$" is just false.

Our, at first rather implausible looking, suggestion is that the generalised truth value of the proposition "$A \in \Delta$" in the macro-state M is to be related to the set of all functions f such that $f(\overline{A}(M))$ *is* a subset of $f(\Delta)$. This condition can be rewritten as $f\circ\overline{A}(M) \subset f(\Delta)$, where $f\circ\overline{A}$ is the function from $\mathcal{S}$ to $\mathbb{R}$ defined by $f\circ\overline{A}(s) := f(A(s))$ for all s in $\mathcal{S}$. We shall denote by $f(A)$ the physical quantity corresponding to the function $f \circ \overline{A}$, and say that $f(A)$ is a *coarse-graining* of A. Then the precise form of our suggestion is that the generalised truth value, $V^M(A \in \Delta)$ of the proposition "$A \in \Delta$" is to be *defined* as the set of all coarse-grainings, $f(A)$, of A for which the

[10]Here, $f(\Delta)$ denotes the set of all real numbers of the form $f(s)$ where s belongs to Δ.

weaker proposition "$f(A) \in f(\Delta)$" *is* true, in the usual sense of 'true'! In symbols, we define the generalised valuation

$$V^M(A \in \Delta) := \{f \mid f(\overline{A}(M)) \subset f(\Delta)\}. \tag{2.3}$$

Of course, it is not obvious that truth values defined in this way have a logical structure; but they do! The proof involves presheaf theory: a subject which we will introduce shortly in the context of finding generalised truth values in quantum theory. Understandably, this involves coarse-graining *operators*, since it is these that represent physical quantities in quantum theory. The use of presheaf ideas in classical physics is discussed in [5].

3 The Presheaf Logic of Quantum Theory

3.1 Coarse-graining in a Quantum Context

The standard, instrumentalist interpretation of quantum theory gives the probability that a proposition "$A \in \Delta$" will be found to be true if measurements[11] are made of the physical quantity A. Specifically, if $|\psi\rangle$ is a normalised state, the probability that the results will lie in the subset Δ of real numbers is

$$\text{Prob}(A \in \Delta;\, |\psi\rangle) = \langle\psi|\, \hat{E}[\, A \in \Delta\,]\, |\psi\rangle \tag{3.1}$$

where $\hat{E}[\, A \in \Delta\,]$ denotes the projection operator onto the subspace of eigenvectors of $\hat{A}$ whose eigenvalues lie in Δ. The operator $\hat{E}[\, A \in \Delta\,]$ is known as a *spectral projector* of the operator $\hat{A}$ that represents the physical quantity A.

A more realist interpretation might aspire to give a truth value to the proposition "$A \in \Delta$" without invoking external measurements. If the quantum state $|\psi\rangle$ is such that $\text{Prob}(A \in \Delta;\, |\psi\rangle) = 1$, it is arguably meaningful to assert that "$A \in \Delta$" is true. Contrariwise, if $\text{Prob}(A \in \Delta;\, |\psi\rangle) = 0$ it might seem natural to say that "$A \in \Delta$" is false, although—motivated by the discussion in Section 2.3 of classical macrostates—one might want to think about situations in which $|\psi\rangle$ is 'close' to a state for which the probability of "$A \in \Delta$" is greater than zero.[12] In any event, in the cases where $0 \leq \text{Prob}(A \in \Delta;\, |\psi\rangle) < 1$ it is certainly not the case that "$A \in \Delta$" is simply true.

One approach would be to define the truth value of "$A \in \Delta$" to *be* the probability $\text{Prob}(A \in \Delta;\, |\psi\rangle)$. This involves the use of *fuzzy logic* in which the truth values of propositions are real numbers in the interval $[0, 1]$. However, we shall adopt a different tack by invoking an operator analogue of the coarse-graining operations used in Section 2.4 in the context of classical physics.

One of the basic structural assumptions in quantum theory is that for any function f, the operator that represents the coarse-grained physical quantity $f(A)$ is $f(\hat{A})$: in this

[11]The plural 'measurements' arises in the relative frequency interpretation of probability. This is the interpretation normally used in instrumentalist approaches to quantum theory.

[12]If $\text{Prob}(A \in \Delta;\, |\psi\rangle) = 1$, might we also want to consider the possibility that $|\psi\rangle$ is 'close' to a state for which the probability is less than 1?

sense, $f(\hat{A})$ is a 'coarse-graining' of the operator $\hat{A}$. Additionally, it is easy to show that the spectral projectors $\hat{E}[\,A \in \Delta\,]$ and $\hat{E}[\,f(A) \in f(\Delta)\,]$ satisfy $\hat{E}[\,A \in \Delta\,] \preceq \hat{E}[\,f(A) \in f(\Delta)\,]$, where $\hat{P}_1 \preceq \hat{P}_2$ denotes that $\hat{P}_1$ projects onto a subspace of the range of $\hat{P}_2$ (*i.e.*, $\mathcal{H}_{P_1}$ is a subspace of $\mathcal{H}_{P_2}$). In this sense, the projection operator $\hat{E}[\,f(A) \in f(\Delta)\,]$ is a coarse-graining of $\hat{E}[\,A \in \Delta\,]$.

Guided by the discussion of classical physics in Section 2.4, one suggestion might be that, for any given quantum state $|\psi\rangle$, the generalised truth value of the proposition "$A \in \Delta$" is the collection of coarse-grainings, $f(\hat{A})$, of $\hat{A}$ such that the weaker proposition "$f(A) \in f(\Delta)$" *is* true—*i.e.*, it is true with probability one, so that $\langle\psi|\,\hat{E}[\,f(A) \in f(\Delta)\,]\,|\psi\rangle = 1$. In other words, we could try the definition (c.f. Eq. (2.3))

$$V^{\psi}(A \in \Delta) := \{f \mid \langle\psi|\,\hat{E}[\,f(A) \in f(\Delta)\,]\,|\psi\rangle = 1\}. \tag{3.2}$$

This possibility arises since $\hat{E}[\,A \in \Delta\,] \preceq \hat{E}[\,f(A) \in f(\Delta)\,]$ implies that, for any f,

$$\langle\psi|\,\hat{E}[\,A \in \Delta\,]\,|\psi\rangle \leq \langle\psi|\,\hat{E}[\,f(A) \in f(\Delta)\,]\,|\psi\rangle \tag{3.3}$$

for all quantum states $|\psi\rangle$. Hence, even if $\langle\psi|\,\hat{E}[\,A \in \Delta\,]\,|\psi\rangle < 1$, there can be functions f such that $\langle\psi|\,\hat{E}[\,f(A) \in f(\Delta)\,]\,|\psi\rangle = 1$.

The use of Eq. (3.2) is perfectly viable, and is discussed in detail in [4]. This includes the construction of the appropriate presheaf needed to show that collections of functions of the type in the right hand side of Eq. (3.2) do have a logical structure.

However, we shall proceed here in a somewhat different way in order to bring out the connection with standard quantum logic. In particular, as explained in Section 2.2, the logical operations 'and', 'or' and 'not', are defined on *projection operators*, not on the underlying propositions. Similarly, the Kochen-Specker theorem deals with the existence of true-false valuations on projectors, not propositions *per se*. This suggests that we should work *ab initio* with projection operators, and hence consider the generalised valuation

$$V^{\psi}(\hat{E}[\,A \in \Delta\,]) := \{f \mid \langle\psi|\,\hat{E}[\,f(A) \in f(\Delta)\,]\,|\psi\rangle = 1\}. \tag{3.4}$$

At this point an important subtlety arises. Namely, it is possible for a pair of propositions "$A \in \Delta$" and "$B \in \Delta'$" to be represented by the *same* projection operator:

$$\hat{E}[\,A \in \Delta\,] = \hat{E}[\,B \in \Delta'\,] \tag{3.5}$$

even if the corresponding operators $\hat{A}$ and $\hat{B}$ do not commute[13]. But then, letting $\hat{P}$ denote $\hat{E}[\,A \in \Delta\,] = \hat{E}[\,B \in \Delta'\,]$, Eq. (3.4) gives the two generalised valuations

$$V^{\psi}(\hat{P}) := \{f \mid \langle\psi|\,\hat{E}[\,f(A) \in f(\Delta)\,]\,|\psi\rangle = 1\} \tag{3.6}$$

$$V^{\psi}(\hat{P}) := \{g \mid \langle\psi|\,\hat{E}[\,g(B) \in g(\Delta')\,]\,|\psi\rangle = 1\} \tag{3.7}$$

and there is no reason why Eq. (3.6) and Eq. (3.7) should be equal.

[13] Of course, something similar happens in classical physics, where many different propositions are represented by the same subset of the state space $\mathcal{S}$. However, the singular feature of quantum theory is that Eq. (3.5) can hold even if $[\,\hat{A}, \hat{B}\,] \neq 0$. The equality in Eq. (3.5) means that $\hat{A}$ and $\hat{B}$ have some simultaneous eigenvectors, but they are not a complete set if $[\,\hat{A}, \hat{B}\,] \neq 0$.

Propositions of this type arise when an operator $\hat{O}$ has vanishing commutators with a pair of operators $\hat{C}, \hat{D}$ with $[\,\hat{C}, \hat{D}\,] \neq 0$. For example, let $\hat{O}$ be the Hamiltonian $\hat{H}$ of the hydrogen atom, and let $\hat{C}$ and $\hat{D}$ be $\hat{L}_x$ and $\hat{L}_y$—the x- and y-components of angular momentum respectively. Then $[\,\hat{H}, \hat{L}_x\,] = 0 = [\,\hat{H}, \hat{L}_y\,]$, and $[\,\hat{L}_x, \hat{L}_y\,] = i\hbar\hat{L}_z \neq 0$. Now, the spectral theorem for commuting operators asserts the existence of hermitian operators $\hat{A}$ and $\hat{B}$ such that $\hat{H}$ and $\hat{L}_x$ are functions of $\hat{A}$, and $\hat{H}$ and $\hat{L}_y$ are functions of $\hat{B}$. Thus, for some set of functions f, g, h, k we have

$$\hat{H} = f(\hat{A}),\ \hat{L}_x = g(\hat{A}) \tag{3.8}$$
$$\hat{H} = h(\hat{B}),\ \hat{L}_y = k(\hat{B}). \tag{3.9}$$

Then, for any[14] subset J of the real numbers, we have[15] $\hat{E}[\,H \in J\,] = \hat{E}[\,f(A) \in J\,] = \hat{E}[\,A \in f^{-1}(J)\,]$, and similarly $\hat{E}[\,H \in J\,] = \hat{E}[\,h(B) \in J\,] = \hat{E}[\,B \in h^{-1}(J)\,]$. Thus $\hat{E}[\,A \in f^{-1}(J)\,] = \hat{E}[\,B \in h^{-1}(J)\,]$, and of course $[\,\hat{A}, \hat{B}\,] \neq 0$ since $[\,\hat{L}_x, \hat{L}_y\,] \neq 0$. Hence this provides an example of the situation envisaged above in regard to Eq. (3.5), with Δ and Δ' chosen to be $f^{-1}(J)$ and $h^{-1}(J)$ respectively.

What this discussion implies is that the truth value assigned to a projection operator $\hat{P}$ should be *contextual*, *i.e.*, it depends on the physical quantity with which one thinks of $\hat{P}$ as being associated. In the example above of the hydrogen atom, with $\hat{P}$ chosen as $\hat{E}[\,H \in J\,]$, the choice is between thinking of this projector as being associated with $\hat{A}$, or with $\hat{B}$. Equivalently, the truth value assigned to the proposition "$H \in J$" depends on whether H is thought of in the context of simultaneously ascribing a truth value to propositions about L_x, or to propositions about L_y.

That such ideas should enter at this point is not surprising since, as remarked earlier, discussions of the physical implications of the Kochen-Specker theorem frequently introduce the notion of contextuality. The important question now is to decide on the most appropriate mathematical framework in which to explore the implications of Eq. (3.6) and Eq. (3.7). There are, in fact, several different (but mathematically equivalent) approaches to this issue, depending on what one decides to call a 'context'.

As indicated above, one choice is to define a context for a projection operator $\hat{P}$ as one of the operators for which it is a spectral projector: this means using the definition in Eq. (3.2) for all physical quantities A and subsets Δ for which $\hat{E}[\,A \in \Delta\,] = \hat{P}$. The mathematical development of this idea involves re-expressing Eq. (3.4) in the language of presheaf theory, and is discussed in [4].

Another possibility is to define a context as an algebra of simultaneously commuting operators to which $\hat{P}$ belongs. In the example of the hydrogen atom, if $\hat{P} = \hat{E}[\,H \in J\,]$, then two such algebras are those generated by $\hat{H}$ and $\hat{L}_x$, and by $\hat{H}$ and $\hat{L}_y$, respectively. This approach is discussed in [6]. However, since the focus of the present paper is logic, we shall use a third possibility, which is to define a context for a projector $\hat{P}$ as a *Boolean algebra* to which $\hat{P}$ belongs. The details are as follows.

[14] Strictly speaking, only *Borel* subsets should be considered, but we will ignore such niceties.

[15] In general, $\hat{E}[\,f(A) \in J\,] = \hat{E}[\,A \in f^{-1}(J)\,]$ for any hermitian operator $\hat{A}$ and (Borel) subset J of real numbers. Here $f^{-1}(J)$ denotes the set of all real numbers s such that $f(s)$ belongs to J.

3.2 Windows on Reality

For each hermitian operator $\hat{A}$, let W_A denote the collection of all projection operators of the form $\hat{E}[\,A \in \Delta\,]$, as Δ ranges over the subsets of the real numbers. This forms a Boolean subalgebra of the non-distributive algebra $\mathcal{L}$ of all projection operators. In addition, for any function f, we have $\hat{E}[\,f(A) \in f(\Delta)\,] = \hat{E}[\,A \in f^{-1}(f(\Delta))\,]$, and hence $\hat{E}[\,f(A) \in f(\Delta)\,]$ belongs to W_A. In this sense, Eq. (3.6) and Eq. (3.7) can be said to assign truth values to the projection operator $\hat{P} = \hat{E}[\,A \in \Delta\,] = \hat{E}[\,B \in \Delta'\,]$ in the *context* of W_A and W_B respectively. In the example of the hydrogen atom, W_A will include the spectral projectors of $\hat{H}$ and $\hat{L}_x$, and W_B will include the spectral projectors of $\hat{H}$ and $\hat{L}_y$.[16]

These remarks suggest that the set, $\mathcal{W}$, of all Boolean sub-algebras of $\mathcal{L}$ is a possible space of contexts in which to assert generalised truth values of projection operators. I shall refer to each such Boolean sub-algebra as a *window* since it gives a partial, Boolean view of the quantum world: a Boolean sub-algebra provides a 'window on reality'.

The next step is to explore the mathematical structure of $\mathcal{W}$. A key property is that it is a partially-ordered set if an ordering (denoted $<$) between windows W_1, W_2 is defined by[17]

$$W_1 < W_2 \quad \text{if} \quad W_2 \subset W_1 \tag{3.10}$$

where the right hand side is to be read as saying that W_2 is a Boolean sub-algebra of W_1 (not just a subset). That this is a partial-ordering[18] is easy to check. From a logical perspective, if $W_2 \subset W_1$ then every element in W_2 can be written as the logical 'or' of disjoint elements in W_1, and hence if $W_1 < W_2$, one can say that W_2 is a *coarse-graining* of W_1. This is represented in Figure 4 where the subsets of the plane bounded by the red and green lines represent the disjoint projectors in W_1 and W_2 respectively.

This ordering on windows is consistent with the $\preceq$-ordering on projection operators in the sense that, for any coarse-graining $f(\hat{A})$ of $\hat{A}$, we have (i) $\hat{E}[\,A \in \Delta\,] \preceq \hat{E}[\,f(A) \in f(\Delta)\,]$; and (ii) $W_{f(A)} \subset W_A$, *i.e.*, $W_A < W_{f(A)}$.

3.3 The Presheaf of Local Truth Values

Because of the Kochen-Specker theorem, binary truth values cannot be assigned consistently to $\mathcal{L}$. However they *can* be assigned to any of its Boolean sub-algebras, W. Such a valuation is a homomorphism from W to the simplest Boolean algebra $\{0, 1\}$, with 0 and 1 being interpreted as 'false' and 'true' respectively.

The next step is to associate with each window W the set, D_W, of all valuations on W. A crucial observation is that if $W_1 < W_2$, there is a map $k_{W_1W_2}$ from D_{W_1} to D_{W_2}. Specifically, let χ be a valuation on W_1: then, since W_2 is a subalgebra of W_1, we can define χ on W_2 by using the values it assign to elements of W_2 considered as members

[16] Equivalently, W_A contains the projectors onto the *simultaneous* eigenstates of $\hat{H}$ and $\hat{L}_x$, and ditto for W_B with $\hat{H}$ and $\hat{L}_y$.

[17] The notation $W_1 < W_2$ includes the possibility $W_1 = W_2$.

[18] This means that (i) for all W we have $W < W$; (ii) $W_1 < W_2$ and $W_2 < W_1$ implies $W_1 = W_2$; and (iii) $W_1 < W_2$ and $W_2 < W_3$ implies $W_1 < W_3$.

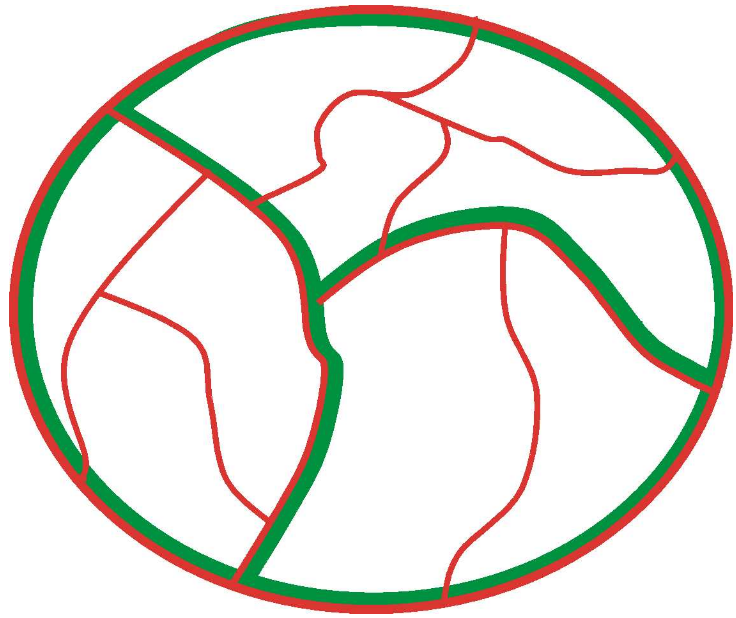

Figure 4: Representation of a situation in which two windows W_1, W_2 satisfy $W_1 < W_2$. The thin (red) lines symbolically enclose elements of the Boolean algebra W_1; the thick (green) lines enclose elements of W_2 that are coarse-grainings of the elements of W_1.

of W_1. These maps $k_{W_1W_2}$ have the property that if $W_1 < W_2 < W_3$ then

$$k_{W_1W_3} = k_{W_2W_3} \circ k_{W_1W_2}. \tag{3.11}$$

This means we have an example of a *presheaf* on the partially-ordered set ('poset') $\mathcal{W}$ of windows. We shall call it the 'presheaf of local truth values'.

To introduce the definition of a presheaf it may be helpful to contrast it with the simpler concept of a *fibre bundle*—something that is much used in modern theoretical physics. A fibre bundle with base space[19] B is an association to each point b in B of a space F_b (the 'fibre over b') with the property that these fibres are all copies of a single space F, known as 'the fibre' of the bundle. The 'bundle space' is then defined to be the union of all the fibres F_b, b in B.

The simplest example of a fibre bundle is a product bundle, defined to be the set of all pairs (b, v) where b is in B, and v in F. Bundles of this type are called 'trivial'. An example is given in Figure 5 where the base space is a circle, S^1, and the fibre has just two points. Figure 6 is an example of a non-trivial bundle with the same fibre and base space. This can be thought of as a Möbius strip with everything but the edges of the strip removed. We see that the fibres 'twist' around as we move round the base space.

An important idea in fibre-bundle theory is that of a *cross-section*. This is defined to be a continuous function from the base space B to the bundle space, with the property that each point b in B is mapped to some point in the fibre F_b over b.[20] For the trivial

[19]The various spaces introduced at this point are all required to be topological spaces. In most applications in theoretical physics they are also differentiable manifolds.

[20]For a product bundle, there is a one-to-one correspondence between cross-sections and maps from

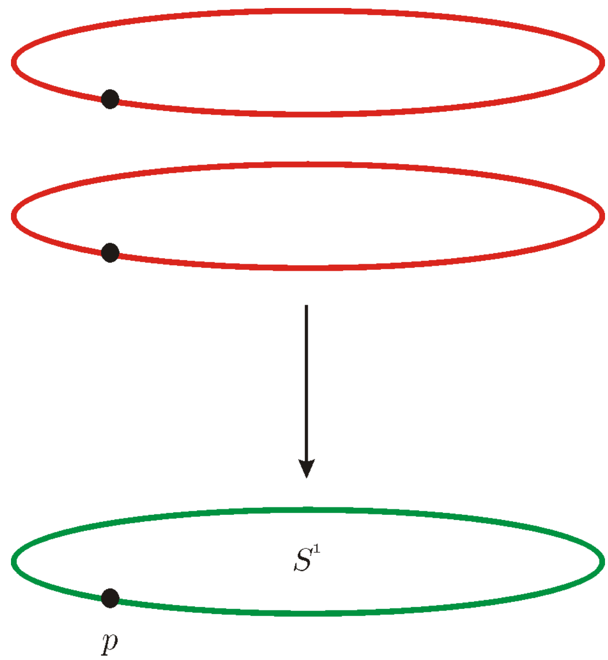

Figure 5: A trivial fibre bundle whose base space is a circle, and whose fibre over each point is a set with two elements.

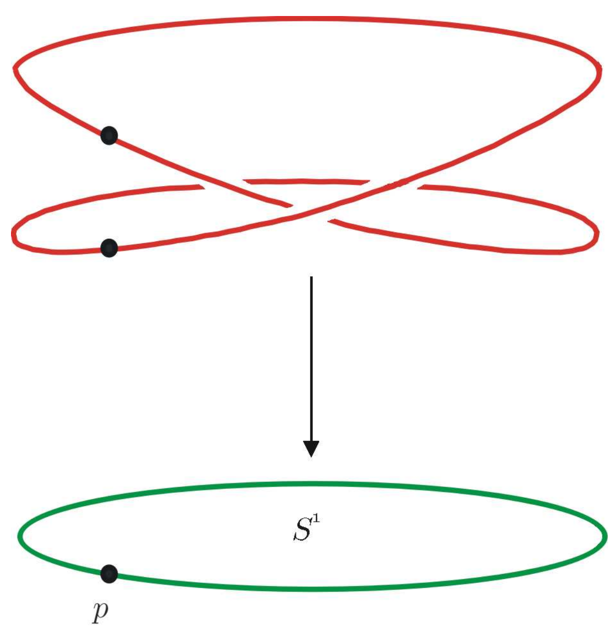

Figure 6: A non-trivial fibre bundle with the same base space as in Figure 5. Each fibre is again a set with two elements, but the bundle has a non-trivial ‘twist’ and is not the same as the bundle space in Figure 5.

bundle in Figure 5 there are just two cross-sections, corresponding to mapping the base space circle into the lower, and upper, circles in the bundle space respectively.

For non-product bundles the situation is different, and there may be no cross-sections at all. For example, this is true of the bundle in Figure 6: any attempt to construct a continuous cross-section inevitably leads to a discontinuity as one works around the base space and comes back to the starting point.

After this preamble, we can return to the idea of a presheaf. A *presheaf*[21] $\mathbf{X}$ over a poset $\mathcal{P}$ is defined to be (i) an association to each p in $\mathcal{P}$ of a space X_p (known as the *stalk* over p); and (ii) an association to each pair p, q such that $p < q$, of a map X_{pq} from X_p to X_q that satisfies the 'coherence' condition that if $p < q < r$ then (c.f. Eq. (3.11))[22]

$$X_{pr} = X_{qr} \circ X_{pq}. \tag{3.12}$$

This is illustrated in Figure 7, where the letters p, q, r, s denote elements in $\mathcal{P}$, and the notation $p \longrightarrow q$ means that $p < q$. Thus a presheaf resembles a fibre bundle except that (i) the stalks at different points in $\mathcal{P}$ need not be copies of a single space (unlike the fibres in a fibre bundle); and (ii) the maps X_{pq} exist when $p < q$.

There is much more to this structure than meets the eye. In particular, presheaf theory can be viewed as a *generalisation* of set theory itself! Specifically, a single set gets replaced by a *parameterised* family of sets X_p, p in $\mathcal{P}$, that are related by the maps X_{pq} from X_p to X_q; this is why a presheaf is sometimes know as a 'varying set'. This generalisation of set theory is very important and is an important part of topos theory [7]. As we shall see, a presheaf embodies a generalisation of the Boolean logic of normal set theory.

Various standard ideas in set theory can be extended to this new context. For example, the analogue of an element x of a set X is a *global element* $\mathbf{x}$ of a presheaf $\mathbf{X}$. This is defined to be an association to each p in $\mathcal{P}$ of a point x_p in X_p such that if $p < q$ then $x_q = X_{pq}(x_p)$. A related idea is a 'partial' element, where the points x_p are defined over only some sub-poset of $\mathcal{P}$. Thus a global element of a presheaf resembles a cross-section of a fibre bundle, and a partial element resembles a bundle section defined on some subset of the base space.

It is not difficult to show that the Kochen-Specker theorem is equivalent to the statement that the presheaf of local truth values has no global elements (although there are partial elements). This is reminiscent of the result that a certain type of fibre bundle has no cross-sections if the bundle is non-trivial; an example is the bundle in Figure 6. Thus we can think of the Kochen-Specker theorem as saying that the presheaf of local truth values is 'twisted' as we move around the space $\mathcal{W}$ of all windows, rather as in Figure 6 the fibres twist around the circle S^1!

B to the fibre F. A cross-section is then analogous to the, so-called, *graph* of a function as discussed in elementary, school-level mathematics.

[21] Actually, this is a very special type of presheaf. In general, a presheaf is defined over a *category*, and a poset is a particularly simple example of a category.

[22] It is also required that, for each p in $\mathcal{P}$, the map X_{pp} is the identity map from X_p to itself.

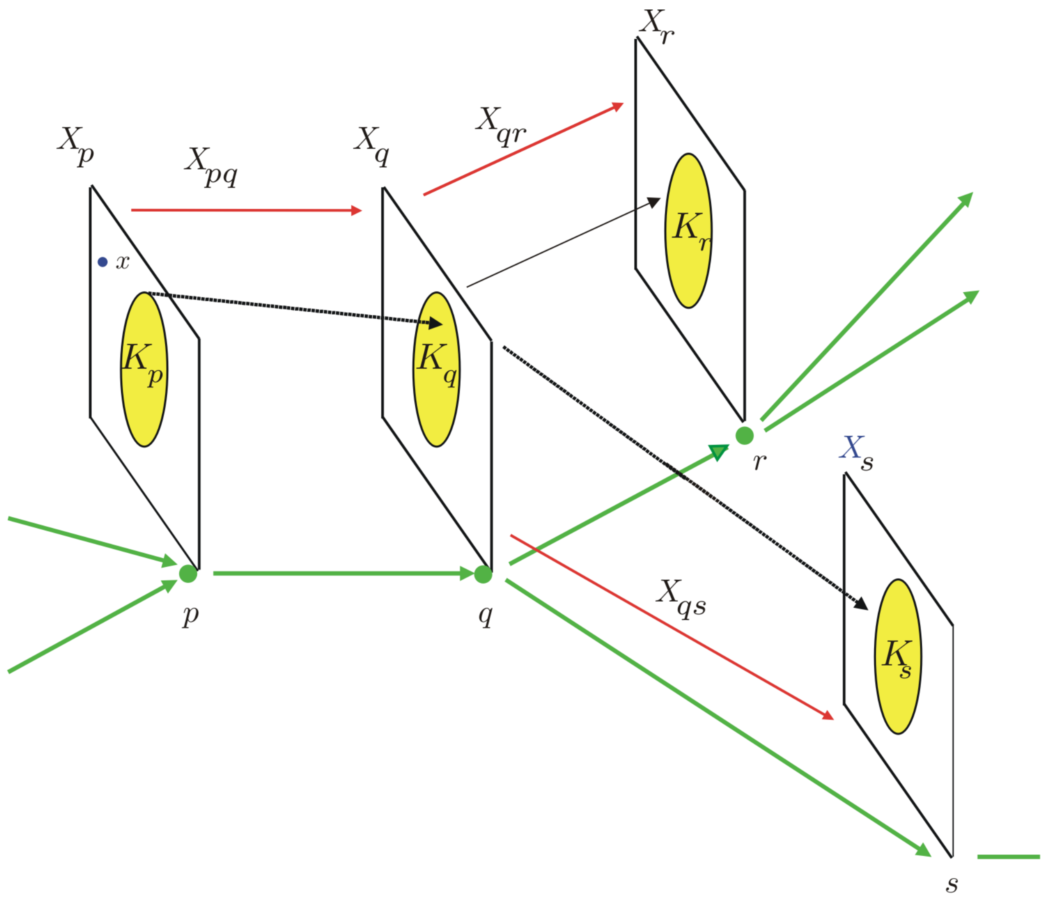

Figure 7: A presheaf $\mathbf{X}$ on a partially ordered set $\mathcal{P}$. This is an association of (i) to each point p in $\mathcal{P}$, a set X_p; and (ii) to each pair of points p, q in $\mathcal{P}$ such that $p < q$, a map X_{pq} from X_p to X_q. These maps satisfy the coherence condition that if $p < q < r$ then $X_{pr} = X_{qr} \circ X_{pq}$. The diagram also illustrates a sub-presheaf $\mathbf{K}$ of $\mathbf{X}$.

3.4 The Presheaf Origin of Contextual Truth Values

Another crucial set-theoretic concept with a presheaf analogue is that of a subset. A *sub-presheaf* (the analogue of a subset) is defined to be an assignment to each p in $\mathcal{P}$ of a subset K_p of X_p with the property that if $p < q$ then $X_{pq}(K_p) \subset K_q$ (c.f. Figure 7). As we shall see, this concept leads to a powerful mathematical way of encoding the idea of 'contextual truth'.

The connection with logic arises in the following way. First consider normal set theory. Then any subset K of a set X is uniquely specified by its characteristic function χ^K on X, defined by

$$\chi^K(x) := \begin{cases} 1 & \text{if } x \text{ is in } K, \\ 0 & \text{otherwise} \end{cases} \tag{3.13}$$

for all x in X. To each subset K, and to each point x in X, there is an associated proposition "$x \in K$", and we can think of χ^K as a valuation of these propositions. Then Eq. (3.13) asserts that the proposition "$x \in K$" is true if and only if x belongs to K: not a terribly surprising result!

However, the presheaf analogue is more subtle, with truth values now being both contextual and multi-valued. This is reflected in the presheaf analogue of a characteristic function. First we need to define the appropriate analogue of the simple set $\{0, 1\}$ of truth values in standard set theory. This involves introducing the concept of a 'sieve at a point p' in $\mathcal{P}$. This is defined to be a subset S of $\mathcal{P}$ such that (i) $p < q$ for all q in S; and (ii) if q belongs to S and $q < q'$, then q' belongs to S. In the quantum case, a sieve on a window W is a collection of coarse-grainings of W such that if some W' belongs to the collection, then so does any coarse-graining of W'.

One of the fundamental results of presheaf theory is that the collection of all sieves at a point p has the structure of a *logic*. Specifically: the operations of 'and' and 'or' on a pair of sieves S_1, S_2 on p are defined to be the intersection $S_1 \cap S_2$, and the union $S_1 \cup S_2$, respectively. These operations are associative and distributive. The operation of negation is more complicated since, if S is a sieve on p, the obvious guess for $\neg S$—the complement of S—is not itself a sieve. Instead, $\neg S$ is the subset of $\mathcal{P}$ defined by

$$\neg S := \{q \mid p < q \text{ and for all } r \text{ with } q < r,\ r \text{ does not belong to } S\} \tag{3.14}$$

which *is* a sieve. With this definition, the collection of all sieves on a point p in $\mathcal{P}$ has the structure of a logic that is almost Boolean, where 'almost' is understood in the following sense.

A Boolean algebra satisfies the famous principle of the excluded middle: namely, a proposition P is always either true or false—in mathematical terms, $P \vee \neg P = 1$, where 1 denotes the proposition that is identically true. For sieves, however, $P \vee \neg P$ is not identically true. Logics of this type are knows as *Heyting algebras* and have been much studied in the topos literature. They can be used to form *deductive systems*, and are hence a bona fide alternative to the familiar logic of daily life.

The relation between sieves and a sub-presheaf is as follows. Let a collection of sets $K_p \subset X_p$, p in $\mathcal{P}$, be a sub-presheaf $\mathbf{K}$ of the presheaf $\mathbf{X}$. Then there is an associated characteristic 'arrow', which, for each p in $\mathcal{P}$, is a map $\chi^{\mathbf{K}}_p$ from X_p to the set of sieves on p, defined by

$$\chi_p^{\mathbf{K}}(x) := \{q \mid p < q \text{ and } X_{pq}(x) \text{ is in } K_q\} \tag{3.15}$$

for all x in X_p (Figure 7 may help at this point). Thus, at p in $\mathcal{P}$, the characteristic arrow assigns to any element x in X_p the set of all points q in $\mathcal{P}$ for which the transformed point $X_{pq}(x)$ *does* belong to the subset K_q of X_q. The definition of a sub-presheaf then implies that the right hand side of Eq. (3.15) is a *sieve* on p.

3.5 The Coarse-Graining Presheaf

When X is a classical state space $\mathcal{S}$, the definition of a characteristic function in Eq. (3.13) provides another way of understanding why the logic of classical physics is Boolean, and with each proposition being either true or false. For, as discussed in Section 2.1, a proposition of the form "$A \in \Delta$" is represented by the subset $\mathcal{S}_{A\in\Delta}$ of $\mathcal{S}$. Then the characteristic function of $\mathcal{S}_{A\in\Delta}$ assigns to a state s the values 1 ('true') if $\overline{A}(s)$ is in Δ, and 0 ('false') if $\overline{A}(s)$ is not in Δ.

On the other hand, as explained in Section 3.1, our suggestion in quantum theory is that the truth value associated with a projection operator $\hat{P}$ should be *contextual*, in the sense that it depends on the window to which one thinks of $\hat{P}$ as belonging. The discussion surrounding Eq. (3.15) then suggests that we might try to define the generalised truth value of a projector $\hat{P}$ in the context of a window W (where $\hat{P}$ belongs to W) to be a sieve of windows on W associated with a sub-presheaf of some presheaf on $\mathcal{W}$.

We might anticipate that this construction should be connected in some way with the type of coarse-graining operation discussed in Section 3.1 in which a projector $\hat{E}[A \in \Delta]$ is replaced with $\hat{E}[f(A) \in f(\Delta)]$. However, at the moment, the only presheaf we have is the presheaf of local truth values, which, since it does not involve coarse-graining, cannot serve for our present purposes.

To proceed further, we return to the ordering operation on $\mathcal{W}$, in which $W_1 < W_2$ means that $W_2 \subset W_1$, and ask how coarse-graining might enter here. More precisely, if $\hat{P}$ is a projection operator in W_1, can it be associated with a projection operator, to be denoted $G_{W_1W_2}(\hat{P})$, that belongs to W_2 and is such that $\hat{P} \preceq G_{W_1W_2}(\hat{P})$?

In fact, typically there are many such projectors in W_2 (Figure 4 may help here), and it is natural to choose the one that is the 'best approximation' to $\hat{P}$, meaning the 'smallest' (the infinum with respect to the $\preceq$-ordering) projector $\hat{Q}$ in W_2 such that $\hat{P} \preceq \hat{Q}$. Hence we define, for all $\hat{P}$ in W_1,

$$G_{W_1W_2}(\hat{P}) := \inf\{\hat{Q} \text{ in } W_2 \mid \hat{P} \preceq \hat{Q}\}. \tag{3.16}$$

Thus $G_{W_1W_2}$ is a map from the Boolean algebra W_1 to the Boolean algebra W_2, and it can be shown that, if $W_1 < W_2 < W_3$, then (c.f. Eq. (3.11) and Eq. (3.12))

$$G_{W_2W_3} \circ G_{W_1W_2} = G_{W_1W_3}. \tag{3.17}$$

It follows that we have constructed a presheaf, denoted $\mathbf{G}$, over $\mathcal{W}$ in which (i) the stalk associated with each window W is a copy of this Boolean algebra, *i.e.*, $G_W := W$; and (ii) the map $G_{W_1W_2}$ from $G_{W_1} = W_1$ to $G_{W_2} = W_2$ is defined by Eq. (3.16).

We shall call $\mathbf{G}$ the *coarse-graining presheaf*, and use it to assign contextual, multi-valued truth values to quantum propositions. The intention is to exploit the notion of a characteristic arrow defined in Eq. (3.15), and, in particular, the fundamental result that the right hand side of this equation is a sieve. Thus the final step is to find the appropriate sub-presheaf of $\mathbf{G}$, and, inspired by Eq. (3.4), we proceed as follows.

For each quantum state $|\psi\rangle$, and each context W, we define the set of 'totally true' projectors in W to be the subset, T^{ψ}_{W}, of projectors $\hat{Q}$ in the Boolean algebra W such that $\langle\psi|\,\hat{Q}\,|\psi\rangle = 1$: *i.e.*, the elements of W to which the quantum formalism assigns a probability of 1.

Next, note that if $\langle\psi|\,\hat{Q}\,|\psi\rangle = 1$ then this is true for any coarse-graining of $\hat{Q}$, *i.e.*, $\langle\psi|\,\hat{Q}\,|\psi\rangle = 1$ implies $\langle\psi|\,\hat{Q}'\,|\psi\rangle = 1$ for all $\hat{Q}'$ such that $\hat{Q} \preceq \hat{Q}'$. This means that the collection of subsets T^{ψ}_{W} of G_W, W in $\mathcal{W}$, forms a *sub-presheaf*, $\mathbf{T}^{\psi}$, of $\mathbf{G}$. Therefore, there is an associated characteristic arrow, defined in Eq. (3.15), and then the theory of presheafs says that the associated (contextual) truth values are sieves, and hence belong to a Heyting algebra.

Rewriting Eq. (3.15) for the special case of the sub-object $\mathbf{T}^{\psi}$ of $\mathbf{G}$ we finally arrive at the following definition of a generalised valuation associated with a quantum state $|\psi\rangle$.[23] The generalised truth value of a projector $\hat{P}$, in the context of the window W to which $\hat{P}$ belongs, is the sieve on W defined by

$$\chi^{\psi}_{W}(\hat{P}) := \{W' \mid W' \subset W \text{ and } \langle\psi|\, G_{WW'}(\hat{P})\,|\psi\rangle = 1\}. \tag{3.18}$$

Thus the truth value, $\chi^{\psi}_{W}(\hat{P})$, associated with a projector $\hat{P}$ is (i) contextual: it depends on the window to we which we think of $\hat{P}$ as belonging; and (ii) multi-valued: $\chi^{\psi}_{W}(\hat{P})$ is a sieve on W, and these form a Heyting algebra.

Note that if the state $|\psi\rangle$ is an eigenvector of $\hat{A}$ with an eigenvalue that lies in Δ, then $\langle\psi|\,\hat{E}[\,A \in \Delta\,]\,|\psi\rangle = 1$, and Eq. (3.18) gives the generalised truth value to be the set of *all* coarse-grainings of W. This is knows as the *principal* sieve on W, and is the unit element of the Heyting algebra of sieves on W.[24]

En passant, we remark that the discussion in Section 2.4 about macro-states in classical physics can be re-expressed in presheaf language, thereby providing a proof that the truth-values are sieves, and hence form a Heyting algebra. The reader is referred to the original papers for further details on these, and related, matters [4, 5, 6].

4 Conclusion

We started by showing how, in classical physics, the idea of coarse-graining can be used to associate a generalised truth value with the proposition "$A \in \Delta$" for a macrostate M even if there are some states s in M for which $\overline{A}(s)$ does not belong to Δ.

Motivated by these ideas we turned to quantum theory, with the aim of using presheaf theory as a natural mathematical framework in which to discuss contextual, multi-

[23] These ideas can be trivially extended to a density matrix state, $\hat{\rho}$, using the fact that $\text{Prob}(P;\hat{\rho}) = \text{tr}(\hat{\rho}\hat{P})$ is the probability associated with the proposition P in the state $\hat{\rho}$.

[24] The null element is the empty set of sieves.

valued, truth values. As the space of contexts we chose the set $\mathcal{W}$ of all Boolean subalgebras of the non-distributive logic of all projection operators. We then constructed two natural presheafs over this space of windows: the presheaf of local truth values, and the coarse-graining presheaf $\mathbf{G}$. For each quantum state $|\psi\rangle$ we constructed a special sub-object, $\mathbf{T}^\psi$ of $\mathbf{G}$, and used this to define the quantum valuation in Eq. (3.18).

In short, we have shown that, notwithstanding the Kochen-Specker theorem, it *is* possible to assign truth-values to the projection operators in a quantum theory, but these truth values are both contextual and multi-valued. It is important to emphasise that the logical connectives ('and', 'or', 'not') are *uniquely* specified by the mathematics of topos theory as applied to presheafs.

However, in addition to being contextual, the presheaf logic differs from a simple Boolean algebra in that it is a *Heyting algebra*, and the principle of excluded middle, $P \vee \neg P = 1$, no longer holds. Equivalently, although it is still true that P implies $\neg\neg P$ it is no longer the case that $\neg\neg P$ implies P. In particular, this means that proofs by contradiction are no longer valid. This is a characteristic feature of, so-called, 'intuitionistic' logic, and (unlike non-distributivity) is easy to live with once one has got used to it.

Does all this mean that, after all, quantum theory *can* be interpreted in a realist way? Clearly the answer is 'no', if 'realist' is understood in the sense used in the Introduction—*i.e.*, propositions about the world are handled using standard Boolean logic. For our truth values are contextual, and multi-valued. On the other hand, the presheaf logic *is* distributive (unlike quantum logic proper) and *can* therefore be used as the basis for a deductive system for reasoning about the world. In this sense, our generalised truth values are closer to classical logic than quantum logic. Jeremy Butterfield and I have referred to the corresponding philosophical position as 'neo-realism'.

At this point, any physicist reader who has courageously slogged through the paper might well say "Well done lads, but is it useful?"—a justified, but frequently embarrassing, question that is routinely addressed to any one claiming to have arrived at a new result in the foundations of quantum theory. One response might be "Well: our way of looking at things gives a better picture, or 'tells a better story', of what quantum theory is saying about the world. And this is valid in its own right".

Personally, I think that this is true (but then I would, wouldn't I?) but nevertheless it would be good to be able to put the scheme to work in some concrete way. The obvious subject area is quantum cosmology, particularly cosmogenesis where the scheme could be used to handle statements about 'how thing are' in that very extreme stage of the universe. In this context it is worth remarking that our scheme can be viewed as a type of 'many-worlds' interpretation of quantum theory, with a 'world' being understood as a 'window on reality': *i.e.*, a Boolean subalgebra of the non-distributive logic of all projectors. The actual working through of this structure in the context of a specific quantum cosmological model remains high on my list of research topics.

A short biography

Chris Isham gained his PhD at Imperial College in 1969 working under the supervision of Paul Matthews. He then spent at year with Abdus Salam at the International Center for Theoretical Physics in Trieste, Italy. In 1970 he started his Lectureship in the

Theoretical Physics group at Imperial College. In 1973 he moved to King's College, London, as a Reader in Mathematics, and returned to Imperial College in 1976. He was appointed to a Chair in 1982.

Chris Isham's main research interests are quantum gravity, and mathematical aspects of the foundations of quantum theory. He also has a deep interest in general philosophy, and the work of C.G. Jung.

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